

NASA Contractor Report 141442

The Determination of Gravity Anomalies from Geoid Heights Using the Inverse Stokes' Formula, Fourier Transforms, and Least Squares Collocation

Reiner Rummel
Lars Sjöberg
Richard Rapp

The Ohio State University
Research Foundation
Department of Geodetic Science
Columbus, Ohio 43212

Prepared Under Contract No. NAS6-2484



National Aeronautics and
Space Administration

Wallops Flight Center
Wallops Island, Virginia 23337
AC 804 824-3411

Foreword

This report was prepared by Dr. Reiner Rummel, Research Associate, Department of Geodetic Science, The Ohio State University (and now at the Deutsches Geodätisches Forschungsinstitut), Dr. Lars Sjöberg, Research Associate, Department of Geodetic Science, and Dr. Richard H. Rapp, Professor, Department of Geodetic Science. This work was supported under NASA Contract No. NAS 6-2484, The Ohio State University Research Foundation Project No. 783904. This contract is administered through the NASA Wallops Flight Center, Wallops Island, Virginia 23337, with Mr. Ray Stanley, Project Scientist.

The three papers have been put together in a single report because of their common goal. Two of the papers are presented here for the first time, but one paper, by Rapp, was presented at the Spring American Geophysical Union Meeting in 1976.

Table of Contents

Foreword	iii
The Determination of Gravity Anomalies from Geoid Heights Using the Inverse Stokes' Formula	1
The Determination of Gravity Anomalies from Geoid Heights Using Fourier Transforms	19
The Determination of Gravity Anomalies from Geoid Heights Using Least Squares Collocation	51

The Determination of Gravity Anomalies from Geoid Heights
Using the Inverse Stokes' Formula

by

Reiner Rummel

Abstract

A numerical method for the determination of gravity anomalies from geoid heights is described using the inverse Stokes formula. This discrete form of the inverse Stokes formula applies a numerical integration over the azimuth and an integration over a cubic interpolatory spline function which approximates the step function obtained from the numerical integration. The main disadvantage of the procedure is the lack of a reliable error measure.

The method was applied on geoid heights derived from GEOS-3 altimeter measurements in the calibration area of the GEOS-3 satellite. The results for the estimation of several 5° , 2° , and 1° mean gravity anomalies are satisfactory when using a regularization cap with a spherical radius of 0.1° and a reference field as e. g. obtained from the GEM 7 potential coefficients up to degree 16.

1. Introduction

The anomalous potential T of the earth's gravity field at a point P on the geoid is linked to the gravity anomaly at the same point via:

$$(1) \quad \Delta g(P) = \left(-\frac{\partial}{\partial h} + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \right) T(P)$$

which is called the fundamental equation of physical geodesy, (c.f. Heiskanen and Moritz, 1967, p. 86). In equation (1), γ is the normal gravity and h the elevation along the normal. The zero order term of the potential is assumed to be zero. In spherical approximation and using Bruns formula to obtain the geoid height N instead of the anomalous potential equation (1) becomes:

$$(2) \quad \Delta g(P) = - \left(\frac{\partial}{\partial r} + \frac{2}{R} \right) G N(P)$$

where $R \dots$ mean radius of the earth, and
 $G \dots$ mean gravity over the earth.

This equation connects the gravity anomaly Δg with the geoid height N and has therefore to be the central equation for all further considerations.

From the boundary condition (2) the solution of the classical problem, the determination of geoid heights from gravity anomalies, is derived the well-known Stokes' equation:

$$(3) \quad N(P) = \frac{R}{4\pi G} \int_{\sigma} S(\psi_{PQ}) \Delta g(Q) d\sigma_Q$$

where $S(\psi_{PQ}) \dots$ Stokes' function for the spherical distance
 $\psi_{PQ} \dots$ between the points P and Q , and
 $\sigma \dots$ unit sphere.

For the solution of the inverse problem, i. e. the determination of the gravity anomaly field from the geoid height, we may use equation (3), too. It is then an integral equation of the first kind. It will be an improperly posed problem in the sense that the unknown gravity anomaly function is discontinuously dependent on the given function $N(P)$. One has to derive a regularized (here better: stabilized) solution of the integral equation (3). One way is to discretize equation (3) and to solve for mean values of a certain block size. Choosing this possibility even an overdetermined system of linear equations can be produced in order to find an approximate solution. For details, see (Gopalapillai, 1974). One may also aim for a global approximation to the gravity anomaly field by minimizing:

$$(4) \quad \min \left\{ \left\| N(P)_{obs} - S \tilde{\Delta g}(Q) \right\|_D^2 + \alpha^2 \left\| \tilde{\Delta g}(Q) \right\|_C^2 \right\}$$

$\begin{matrix} n & & n & \infty \\ 1 & & \infty & 1 \end{matrix} \qquad \qquad \qquad \begin{matrix} \infty \\ 1 \end{matrix}$

The Stokes equation (3) is written in (4) in operator form. With expression (4) we minimize on one hand the discrepancy vector between the n given geoid heights and the corresponding geoid heights derived from the integration over the approximate gravity anomaly field $\tilde{\Delta g}(Q)$, where " \sim " indicates "estimate". D is the error variance-covariance matrix with dimension $(n \times n)$. On the other hand, (4) tries to obtain the smoothest approximation $\tilde{\Delta g}(Q)$, smoothest in the sense of minimum norm, where C is the global covariance function of the gravity anomaly field, e.g. the one derived by Tscherning and Rapp (1974). In expression (4) α is the regularization - (stabilization) - parameter that balances the first term against the second. The global character of the approximation should be indicated by the dimension ∞ . The solution of the minimum problem (4) is:

$$(5) \quad \tilde{\Delta g}(Q) = C^{-1} S^T (S C^{-1} S^T + \alpha D)^{-1} N(P)_{obs}$$

$\begin{matrix} \infty \\ 1 \end{matrix} \qquad \begin{matrix} \infty & \infty & n & \infty & \infty \\ \infty & n & \infty & \infty & n \end{matrix} \qquad \begin{matrix} n \\ n \end{matrix} \qquad \begin{matrix} n \\ 1 \end{matrix}$

which is for $\alpha = 1$ identical to the least-squares collocation solution. It was successfully applied to this problem in (Rapp, 1974) and (Rummel and Rapp, 1977).

Departing from the boundary condition (3) there exists alternatively to the above shown integral equation approach a direct inverse formulation of our problem. It provides a direct means for the determination of gravity anomalies from geoid heights and was first given in (Molodenskii et al., 1962):

$$(6) \quad \Delta g(P) = -\frac{G}{R} N(P) - \frac{G}{16\pi R} \int_{\sigma} \frac{N(Q) - N(P)}{\sin^3 \frac{\psi_{PQ}}{2}} d\sigma_P$$

The derivation of equation (6) from the boundary condition (2) will be given in the next section. Equation (6) could also be considered as a solution to the Stokes formula (3) posed as an integral equation. Equation (6) again poses a regularization problem since for $\lim \psi_{PQ} \rightarrow 0$ a small error δ in $N(Q)$ will cause a large error ϵ in the determined gravity anomaly.

This report will describe the discrete treatment of equation (6) given geoid heights as derived from altimeter observations. Results for estimated gravity anomalies computed from geoid heights will be presented as obtained from GEOS-3 altimeter data in the calibration area of GEOS-3.

Before we continue to discuss the numerical treatment of equation (6) it should be mentioned that gravity anomaly recovery using the integral equation approach, equation (3), as well as the inverse Stokes formula, equation (6), may also be tackled in the spectral domain. This means development into spherical harmonics for a global solution on the sphere. The spectral form of equation (3) and of equation (6) are identical when expressed in spherical harmonics, namely:

$$(7) \quad g_{nm} = \frac{G}{R} (n-1) t_{nm}$$

where

g_{nm} ... spherical harmonic coefficient of the gravity anomaly field of degree n and order m ,
 t_{nm} ... spherical harmonic coefficient of the geoid height.

The regularized form derived from equation (5) takes the form:

$$(8) \quad g_{nm} = \left[1 + \frac{\alpha \sigma_o^2}{c_n s^{n+1}} \left(\frac{G}{R} (n-1) \right)^2 \right]^{-1} \frac{G}{R} (n-1) t_{obs, nm}$$

with

σ_o^2 variance of the observation error
 c_n degree variance of the gravity anomaly field
 $s = \left(\frac{R_{Bje}}{R} \right)^2$... square ratio of the radius of a chosen Bjerhammar sphere to the mean radius of the earth

For the discrete form of equations (3) or (6), the analysis in the spectral domain means eigenvalue analysis of the corresponding system of linear equations. This will be more or less done for the discrete treatment of equation (6) as shown in the sequel. Finally, when using a planar approximation of equations (3) and (6), a Fourier analysis could be performed. Again, the spectral form of the planar of equations (3) and (6) will be identical in the spectral domain.

2. Formulation of the Inverse Stokes Formula for Discrete Data

As already mentioned, we will shortly derive the inverse Stokes formula from the boundary value condition (2) which is in essence a spherical approximation of equation (1):

$$(2) \quad \Delta g(P) = -G \left(\frac{\partial N(P)}{\partial r} + \frac{2}{R} N(P) \right)$$

Provided $N(Q)$ is twice differentiable for $\lim Q \rightarrow P$ the radial derivative $\frac{\partial N(P)}{\partial r}$ becomes, using the Poisson integral (Heiskanen and Moritz, 1967, p. 38):

$$(9) \quad \frac{\partial N(P)}{\partial r} = -\frac{1}{R} N(P) + \frac{R^2}{2\pi} \int_{\sigma} \frac{N(Q) - N(P)}{\ell_o(P-Q)^3} d\sigma_q$$

with $\ell_o \dots$ spatial distance between P and Q . Inserting equation (9) into equation (2), we obtain with:

$$\ell_o(P-Q) = 2R \sin \frac{\psi_{PQ}}{2}$$

the inverse Stokes formula

$$(6) \quad \Delta g(P) = -\frac{G}{R} \left(N(P) + \frac{1}{16\pi} \int_{\sigma} \frac{N(Q) - N(P)}{\sin^3 \frac{\psi_{PQ}}{2}} d\sigma_q \right)$$

The geoid heights derived from altimeter measurements are not given as a continuous function covering the sphere that approximates the earth as it would be required when using equation (6). Instead, there is only available:

- a finite amount of data
- arranged in arc segments with a certain along track and cross track spacing, and
- disturbed by a certain amount of error.

Thus we have to aim for a discrete solution of equation (6). The quality of the discrete formulation of the problem will entirely depend on the quality with which the radial derivative $\frac{\partial N(P)}{\partial r}$ in equation (9) can be approximated from discrete data.

The numerical treatment of the type of singular integral appearing on the right side of equation (9) is discussed in (Meissl, 1971). The procedure described there for the regularization of the integral cannot be applied in our case, not only because of the large computational effort that is necessary for its use, but mainly because of a lack of data dense enough to make this procedure applicable.

Integration Over the Azimuth

Separating the terms dependent only of the spherical distance ψ from those also dependent of the azimuth, equation (6) is rewritten to:

$$\Delta g(P) = -\frac{G}{R} \left(N(P) + \frac{1}{16\pi} \int_{\psi} \frac{\sin \psi_{PQ}}{\sin^3 \frac{\psi_{PQ}}{2}} \int_{\alpha} (N(Q) - N(P)) d\alpha_Q d\psi_Q \right)$$

with

$$\sin \psi = 2 \sin \frac{\psi}{2} \cos \frac{\psi}{2}$$

and with

$$(10) \quad \int_{\alpha} (N(Q) - N(P)) d\alpha_Q = 2\pi (\overline{N(\psi_{PQ})} - N(P)) = 2\pi \delta N(P, \psi_{PQ})$$

we obtain

$$(11) \quad \Delta g(P) = -\frac{G}{R} \left(N(P) + \frac{1}{4} \int_{\psi} \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} \delta N(P, \psi_{PQ}) d\psi_Q \right)$$

The integration over the azimuth can be considered as an averaging process that yields a function of $\delta N(P, \psi_{PQ})$ for the computation point P only dependent of the spherical distance ψ . As we know the accuracy of such an averaging process will be hardly affected by few bad data as long as many data enter the computation. For the numerical integration, the infinitesimal increment $d\psi$ with spherical radius ψ_Q from the computation point is replaced by the finite increment $\Delta\psi$, with e. g. $\Delta\psi = 0.5^\circ$. These circular concentric zones are subdivided into four subblocks, as shown in Figure 1 with a certain arrangement of ground tracks of given geoid heights. A subdivision by four is chosen to obtain a representative mean value for each circular zone from the four subblocks even for a nonhomogeneous coverage of the circular zone by altimeter tracks. A higher subdivision could not warrant that each subblock is covered by an altimeter track.

First we average the given geoid heights in the four subblocks, then the four mean values are averaged to provide a mean value of the circular zone. This procedure yields a step function with step width $\Delta\psi$ with the mean values of all circular zones which approximates the function $\delta N(P, \psi_{PQ})$ of equation (10), as shown in Figure 2.

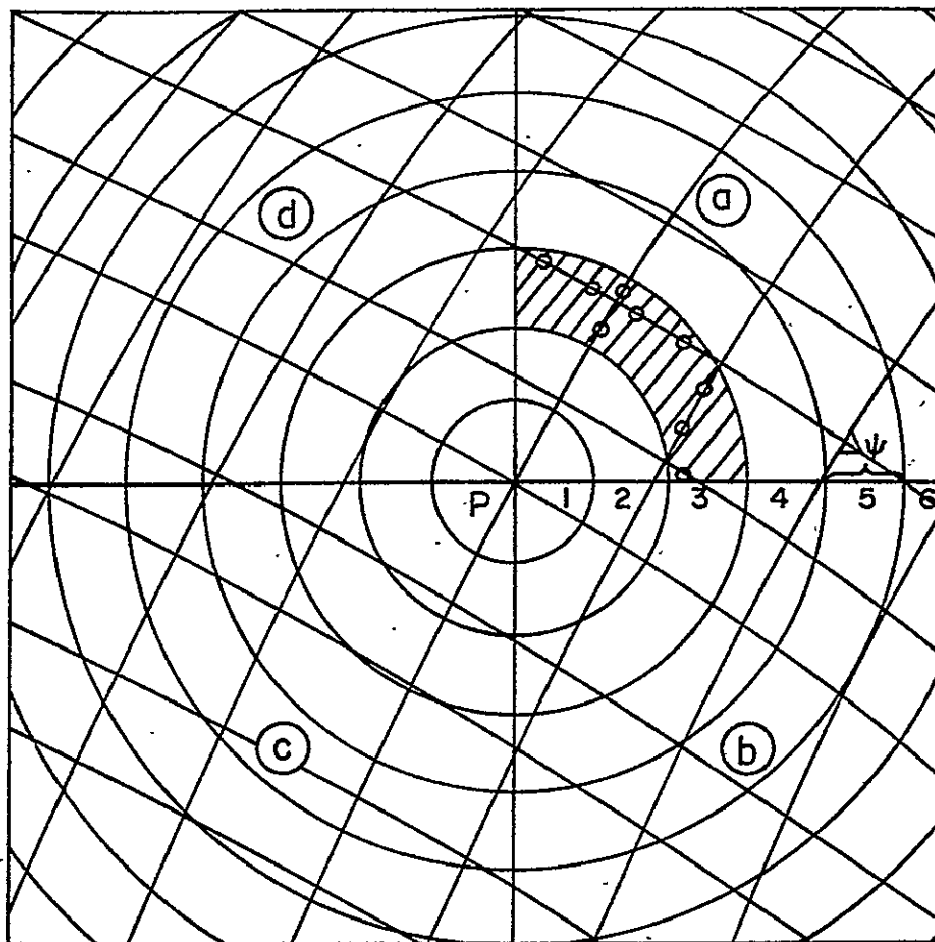


Figure 1: The template of circular zones around the computation point P: As an example, all geoid heights (marked with "o") inside segment "a" of circular zone 3 will be averaged.

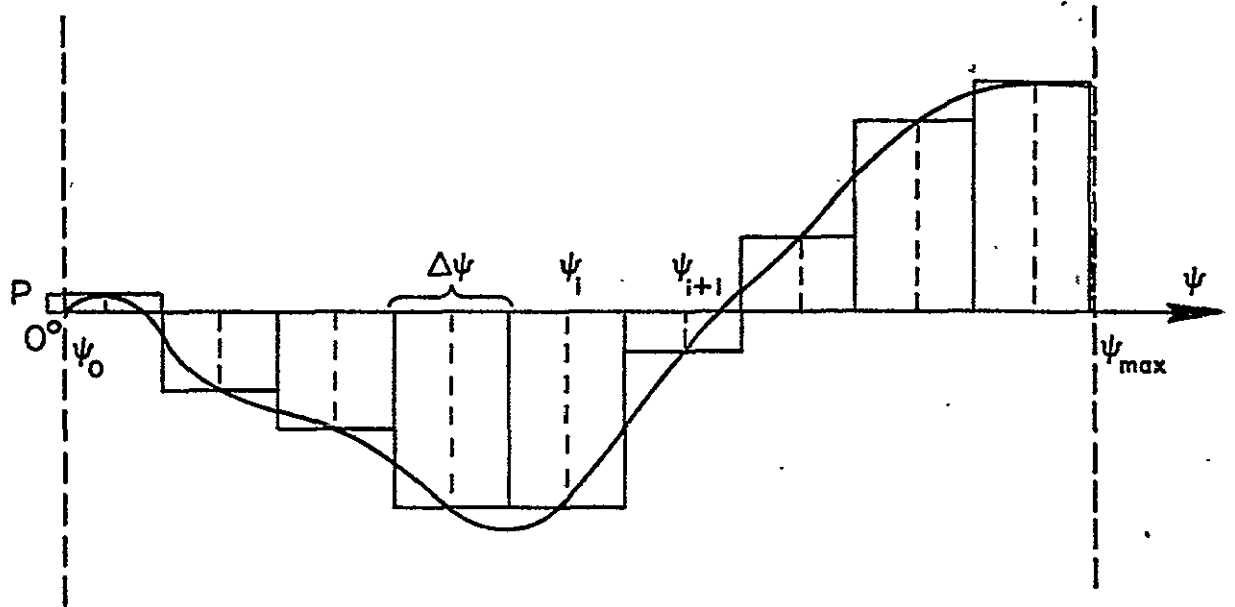


Figure 2: Step function derived from the integration over the azimuths and cubic spline function approximating the step function. The integration over ψ is carried out from ψ_0 to ψ_{\max} .

Unfortunately, in the vicinity of the singularity point P, the circular zones are so small that only few geoid heights will be given inside them to be used for the average process. This may result in large errors for the mean values and may cause large errors in the determination of the gravity anomalies. A stabilization is therefore necessary to maintain sufficient accuracy at the expense of resolution, as will be described later.

Integration Over the Spherical Distance

We approximate the step function by an interpolatory polynomial function that attains the mean value of the i-th step (i.e. the mean value of the i-th circular zone with inner radius $\psi_i - \frac{\Delta\psi}{2}$ and outer radius $\psi_i + \frac{\Delta\psi}{2}$) at the midpoint of the i-th step with spherical distance ψ_i from the computation point P, as shown in Figure Two. This means the polynomial function:

$$(12) \quad p(\psi) = \sum_{j=1}^n a_j \bar{\Phi}_j(\psi)$$

where

a_j polynomial coefficient, and
 $\bar{\Phi}_j(\psi)$... base functions

fulfills the condition

$$(13) \quad \delta N(P, \psi_i) = p(\psi_i)$$

We insert expression (12) into equation (11) and obtain:

$$(14) \quad \Delta g(P) = -\frac{G}{R} \left(N(P) + \frac{1}{4} \sum_{j=1}^n a_j \int_{\psi}^{\psi} \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} \bar{\Phi}_j(\psi) d\psi \right)$$

Since only the coefficients a_j of the interpolatory series depend on the values of the step function the integration over ψ can be performed once for all with highest possible precision:

$$(15) \quad I_j = \frac{1}{4} \int_{\psi}^{\psi} \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} \bar{\Phi}_j(\psi) d\psi$$

The gravity anomaly determination becomes, using equations (14) and (15):

$$(16) \quad \Delta g(P) = -\frac{G}{R} \left(N(P) + \sum_{j=1}^m a_j I_j \right)$$

For the interpolatory function a cubic interpolatory spline function is chosen. Assuming n function values $\delta N(P, \psi_i)$ to be given, it has the form:

$$(17) \quad \begin{aligned} p_i(\psi) = & \frac{M_i}{6\Delta\psi} (\psi_{i+1} - \psi)^3 + \frac{M_{i+1}}{6\Delta\psi} (\psi - \psi_i)^3 + \\ & + \left(\frac{\delta N(P, \psi_{i+1})}{\Delta\psi} - \frac{M_{i+1}\Delta\psi}{6} \right) (\psi - \psi_i) + \\ & + \left(\frac{\delta N(P, \psi_i)}{\Delta\psi} - \frac{M_i\Delta\psi}{6} \right) (\psi_{i+1} - \psi), \quad i = 0, 1, \dots, n-1 \end{aligned}$$

with

$$\Delta\psi = \psi_{i+1} - \psi_i, \text{ and}$$

$$M_i = p''(\psi_i), \text{ i. e. the second derivative of}$$

$$p(\psi) \text{ at } \psi = \psi_i$$

The advantage of using cubic splines is that the coefficients M_i can be derived from a simple recursion formula, (cf. Shampine and Allen, 1973, p. 54). For the unique definition of $p(\psi)$ we define $M_0 = M_{n-1} = 0$. The function $p(\psi)$ defined that way fulfills the condition of being twice differentiable for $\lim \psi \rightarrow 0$, which is required for the function $\delta N(P, \psi)$ when using equation (9).

Employing the basic principle of equation (16) and scaling $\Delta\psi = 1$, we define:

$$(18a) \quad I_{1,i} = \frac{1}{24} \int_{\psi_i}^{\psi_{i+1}} [(\psi_{i+1} - \psi)^3 - (\psi_{i+1} - \psi)] \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} d\psi$$

$$(18b) \quad I_{2,i} = \frac{1}{24} \int_{\psi_i}^{\psi_{i+1}} [(\psi_i - \psi)^3 - (\psi_i - \psi)] \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} d\psi$$

$$(18c) \quad I_{3,i} = \frac{1}{4} \int_{\psi_i}^{\psi_{i+1}} (\psi_{i+1} - \psi) \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} d\psi, \text{ and } \dots$$

$$(18d) \quad I_{4,i} = \frac{1}{4} \int_{\psi_i}^{\psi_{i+1}} (\psi_i - \psi) \frac{\cos \frac{\psi}{2}}{\sin^2 \frac{\psi}{2}} d\psi$$

The integration is carried out numerically in extremely small steps by Gaussian quadrature. Inserting equations (18, a - d) into equation (11), the estimation formula for a gravity anomaly at a point P becomes:

$$(19) \quad \Delta \tilde{g}(P) = -\frac{G}{R} \left(N(P) + \sum_{i=0}^{n-1} [M_i I_{1,i} - M_{i+1} I_{2,i} + \right. \\ \left. + \delta N(P, \psi_i) I_{3,i} - \delta N(P, \psi_{i+1}) I_{4,i}] \right)$$

In equation (11) the integration ranges from $\psi = 0^\circ$ to $\psi = 180^\circ$ which would require data given over the entire globe. Actually, because of the rapid decrease of the integral kernel $\frac{1}{\sin^3 \frac{\psi}{2}}$ in the inverse Stokes formula, accepting a small error,

the integration can be truncated at a suitable ψ_{\max} . Gopalapillai (1974) investigated the error of truncation. From his results we conclude that an integration up to $\psi_{\max} = 10^\circ$ suffices at the present accuracy level for the geoid heights. This means that we accept an error of $\sigma_{tr} \pm 0.3$ mgal for the estimated gravity anomalies when a reference field up to degree $n_{\max} = 16$ is introduced and an error of $\sigma_{tr} = \pm 3$ mgal for a reference field with $n_{\max} = 4$.

The integration from $\psi = 0^\circ$ causes, as already noted, stability problems. At the critical point $\psi = 0^\circ$, the integral kernel goes to infinity with the strength $\frac{1}{\sin^3 \frac{\psi}{2}}$, whereas for the difference $\delta N(P, \psi_{PQ})$ we know only it has to be twice

differentiable. A small error δ in the geoid height difference close to the singularity point will therefore blow up to a large error ϵ in the estimated gravity anomaly. Because of the lack of dense data, the regularization method proposed by Meissl (1971) cannot be employed in our case, so we introduce a very pragmatic method of regularization. We postulate that the geoid height difference $\delta N(P, \psi_{PQ})$ is zero in a small spherical cap around P with spherical radius ψ_0 . In our computations, the radius of the "regularization cap" was varied between $\psi_0 = 0.001^\circ$ and $\psi_0 = 0.1^\circ$, which corresponds to a distance of approximately 0.2 km to 22 km. The integration is thus actually carried out from $\psi = \psi_0$ to ψ_{\max} .

The inverse Stokes formula, equation (6), provides point values at $P \in \sigma$. What we are interested in are mean values of e.g. block size $1^\circ \times 1^\circ$ or $2^\circ \times 2^\circ$. The correct determination of mean values of a block would require an integration over the block or in other words, the geoid height function given continuously inside this block. The procedure used in our computations first estimates all point values of gravity anomalies inside a block for which discrete geoid heights of an arc segment are given, using equation (19). Then these point estimates are simply averaged to give the mean gravity anomaly of the block.

As a severe drawback of the use of the inverse Stokes formula in discrete form, we have to accept the fact that there is no easy way to find a reliable error estimate to the estimated mean gravity anomalies.

3. Numerical Results

The data used in the computations were geoid heights as derived from GEOS-3 altimeter measurements in the calibration area of the GEOS-3 satellite. The distribution of the source data and the method of the geoid height determination are described in (Rummel and Rapp, 1977). We estimated mean gravity anomalies for the same blocks where the mean gravity anomaly recovery was performed using the least squares collocation technique, equation (5), and also described in (Ibid., p. 82). These are three 5° equal area anomalies, two $2^\circ \times 2^\circ$ anomalies, and two $1^\circ \times 1^\circ$ anomalies, as shown in Figure Three. The location of these blocks is chosen to maximize the number of altimeter tracks given at that time in and around the blocks.

The computations were carried out with the geoid heights referred to an ellipsoidal reference field and with the geoid heights referred to a reference field up to degree $n_{\max} = 16$ computed from the coefficients of the Goddard Earth Model 7, (Wagner et al., 1975), using:

$$(20) \quad N(P)_{\text{GEM 7}} = \frac{GM}{\gamma R} \sum_{n=2}^{n_{\max}} \sum_{m=0}^n (\bar{C}_{nm}^* \cos m\lambda_P + \bar{S}_{nm} \sin m\lambda_P) P_{nm}(\sin \varphi_P)$$

for the computation of the reference geoid heights. All terms of equation (20) are explained in (Rapp and Rummel, 1975, p. 2). We have then to add to the estimated residual gravity anomaly a reference gravity anomaly also computed from the same set of potential coefficients up to degree n_{\max} using equations (20) and (7).

The results of the gravity anomaly recovery using the discrete form of the inverse Stokes formula are shown in Table One, Columns 7 to 11. They are given for different regularization caps ($\psi_0 = 0.001^\circ, 0.01^\circ, 0.1^\circ$) and for an ellipsoidal as well as for a low degree ($n_{\max} = 16$) reference field. In column 12

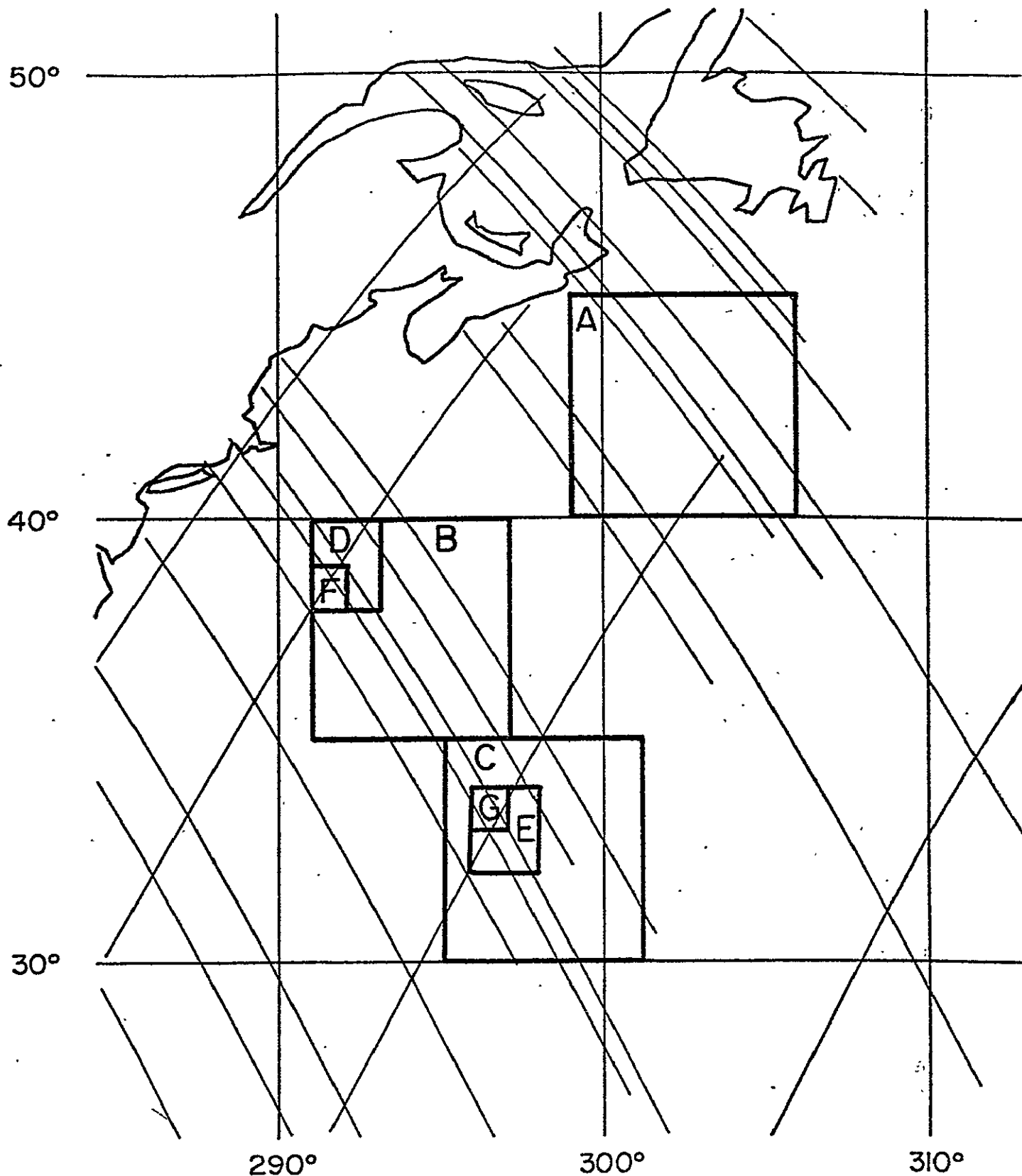


Figure 3. Location of Test Blocks and Related Altimeter Tracks in the Anomaly Estimation Area

the number of point values computed to obtain the mean gravity anomalies are given. Table One gives also the results of the mean gravity anomaly recovery when using the least squares collocation method, taken from (Rummel and Rapp, 1977, Table Five). In column 4 the known terrestrial mean gravity anomalies are shown for the same blocks; in column 13 the mean gravity anomalies as derived solely from the potential coefficients of the GEM 7 earth model up to $n_{\max} = 16$ are given.

4. Conclusions

The results of Table One for the estimation of mean gravity anomalies using the inverse Stokes formula show:

--- that a regularization cap with a too small spherical radius ψ_0 ($\psi_0 = 0.001^\circ$ and also $\psi_0 = 0.01^\circ$) yields too unstable results. Only for a $\psi_0 = 0.1^\circ$ stable estimates could be obtained.

--- only the use of the reference field (here $n_{\max} = 16$) could provide satisfactory results, whereas the gravity anomaly estimates using the ellipsoidal reference field were to far off in comparison with the terrestrial gravity anomalies.

Using the reference field (GEM 7, $n_{\max} = 16$) and a regularization cap with $\psi_0 = 0.1^\circ$ the presented method for the use of a discrete form of the inverse Stokes formula produced satisfactory results. The results, as shown in column 11 of Table One, are in good agreement with the given terrestrial gravity anomalies (column 4) and with the results obtained from collocation (columns 5 and 6). Thus, the described method can be considered a valuable alternative to the least squares collocation method.

Its advantage in comparison with collocation will be that it needs less computer time; no system of linear equations has to be solved. The drawbacks are:

--- there is no easy way to find a reliable error estimate to the determined mean gravity anomalies.

--- the method of regularization is from the theoretical point of view hardly satisfactory.

For a further improvement of the presented method, one should therefore investigate more appropriate but still economical methods of regularization. In addition, instead of the type of spline function used for these computations, one should use a mean value reproducing spline, as described in (Sünkel, 1975) which, when integrated over the intervals $\Delta\psi$, yields exactly the ordinate value of the step function. More stable results could be obtained using an adjusting spline function with less nodes than given steps of the step function. Finally, we would expect to obtain better gravity anomaly recovery results given denser geoid height data without significantly increasing the computer time.

Table One: Results of the estimation of mean gravity anomalies using the inverse Stokes formula

1	2	3	4	5	6	7	8	9	10	11	12	13
block	size	coordinates φ λ	terr. Δg	collo- cation* (ellip.)	collo- cation* (GEM 7 (12, 12))	inv. Stokes (ellip.) $\psi_0 =$ 0.001°	inv. Stokes (GEM 7 (16, 16)) $\psi_0 =$ 0.001°	inv. Stokes (ellip.) $\psi_0 =$ 0.01°	inv. Stokes (GEM 7 (16, 16)) $\psi_0 =$ 0.01°	inv. Stokes (GEM 7 (16, 16)) $\psi_0 =$ 0.1°	# of point values	Δg from (GEM 7 (16, 16))
A	5° E. A.	45°- 40° 299°-306°	-17 ± 3	-16 ± 6	-20 ± 3	-23	-31	-13	-26	-21	113	-15
B	5° E. A.	40°- 35° 291°-297°	-24 ± 2	-24 ± 5	-25 ± 3	-10	-31	- 7	-27	-24	93	-16
C	5° E. A.	35°- 30° 295°-301°	- 8 ± 3	- 3 ± 6	-11 ± 3	12	14	11	-14	-15	97	-23
D	2° x 2°	40°- 38° 291°-293°	-31 ± 7	-33 ± 7	-32 ± 5	- 9	-41	-10	-39	-36	26	-22
E	2° x 2°	34°- 32° 296°-298°	- 8 ± 7	- 2 ± 7	- 8 ± 5	11	- 6	10	- 7	- 5	23	-13
F	1° x 1°	39°- 38° 291°-292°	-25 ± 14	-30 ± 11	-29 ± 9	-27	-38	-21	-33	-29	8	-12
G	1° x 1°	34°- 33° 296°-297°	- 8 ± 15	- 3 ± 10	- 9 ± 8	6	-13	7	-12	-14	9	-22

* from (Rummel and Rapp, 1977)

References

- Gopalapillai, S., Non-Global Recovery of Gravity Anomalies from a Combination of Terrestrial and Satellite Altimetry Data, Department of Geodetic Science Report No. 210, The Ohio State University, Columbus, 1974.
- Heiskanen, W. and H. Moritz, Physical Geodesy, W. H. Freeman Co., San Francisco, 1967.
- Meissl, P., Preparations for the Numerical Evaluation of Second Order Molodensky-Type Formulas, Department of Geodetic Science Report No. 163, The Ohio State University, Columbus, 1971.
- Molodenskii, M. S., V. F. Eremeev and M. I. Yurkina, Methods for Study of the External Gravitational Field and Figure of the Earth, (transl. from Russian), Israel Program for Scientific Translations, Jerusalem, 1962.
- Rapp, R. H., Gravity Anomaly Recovery from Satellite Altimetry Data Using Least Squares Collocation Techniques, Department of Geodetic Science Report No. 220, The Ohio State University, Columbus, 1974.
- Rapp, R. H. and R. Rummel, Methods for the Computation of Detailed Geoids and their Accuracy, Department of Geodetic Science Report No. 233, The Ohio State University, Columbus, 1975.
- Rummel, R. and R. H. Rapp, Undulation and Anomaly Estimation Using GEOS-3 Altimeter Data without Precise Satellite Orbits, Bulletin Geodesique, 51, 1, 73-88, 1977.
- Shampine, L. F. and R. C. Allen, Numerical Computing: An Introduction, Saunders Company, Philadelphia, 1973.
- Sünkel, H., Reconstruction of Functions from Discrete Mean Values Using Cubic Spline Functions, in : Contributions of the Graz Group to the XVI General Assembly of IUGG/IAG in Grenoble 1975, eds: P. Meissl, H. Moritz and K. Rinner, Mitteilungen der geodätischen Institute der Technischen Universität Graz, Folge 20, Graz, 1975.
- Tscherning, C. C. and R. H. Rapp, Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree Variances, Department of Geodetic Science Report No. 208, The Ohio State University, Columbus, 1974.
- Wagner, C. A., F. J. Lerch, J. E. Brown and J. A. Richardson, Improvement in the Geopotential Derived from Satellite and Surface Data (GEM 7 and 8), Journal of Geophysical Research, 82, 5, 901-914, 1977.

The Determination of Gravity Anomalies from Geoid Heights
Using Fourier Transforms

by

Lars Sjöberg

PRECEDING PAGE BLANK NOT FILLED

Abstract

A method is outlined for the prediction of mean free-air gravity anomalies from altimeter data along one arc at a time. The method may be regarded as least squares collocation performed in the frequency domain. In this way a considerable gain in the necessary computer time is achieved. The final prediction of a block mean anomaly is estimated by the mean value of all separate estimates from different arcs. A simple formula is derived for estimating the prediction errors..

The method is tested for the prediction of 5° equal area anomalies and $1^\circ \times 1^\circ$ anomalies in the GEOS-3 Calibration Area and in the Philippines Area. For example, approximately 12800 observations in 85 arc segments were processed for the prediction of 6 5° anomalies in the Calibration Area. The total computer time was 2.1 minutes and the RMS difference between predicted and terrestrial anomalies was 4 mgal. In the same area a subset of 21 $1^\circ \times 1^\circ$ anomalies was predicted from 41 arc segments (approximately 6500 observations) with a corresponding RMS difference of 5 mgal.

Due to the simplicity of the method, all available altimeter arcs in an area can easily be included. Additional information may in a simple way be taken into account in a posterior prediction.

1. Introduction

The method of least squares filtering of data measured at discrete points along a profile was presented by H. Moritz (1967). In 1969, Moritz extended the method to the prediction in the area between observed profiles. The latter technique suffers from the disadvantage of requiring parallel profiles of observations. However, Moritz' idea of predicting in points not on a profile may be generalized to the prediction of gravity anomalies from altimeter data given along one profile (arc) at a time. Then the final estimates are determined by the mean value estimates provided by all the arcs in the area.

The method may be regarded as least squares collocation performed in the frequency domain (instead of in the covariance domain). Subsequently, the covariance functions are transformed to the corresponding spectra. The autocovariance matrix is transformed to diagonal form, which considerably reduces the time of computation. Thus the very large amount of available altimeter information can be handled in a simple way.

2. Theoretical Background

Let us assume that a stochastic process x is measured at n points (discrete case). The process y related to x can then be estimated linearly by the observations x_1 in the following way:

$$(2.1) \quad \hat{y}_P = \sum_{i=1}^n h_{P1} x_1$$

where h_{P1} are the weights of the observations. The prediction error is given by:

$$\epsilon_P = \hat{y}_P - y_P = \sum_{i=1}^n h_{P1} x_1 - y_P$$

The linear least squares prediction problem is to estimate the weights in such a way that the RMS prediction error is a minimum. We assume that the statistical expectations of x and y are 0:

$$(2.2) \quad E\{x\} = 0 \quad \text{and} \quad E\{y\} = 0$$

For each set of weights the prediction variance becomes:

$$(2.3) \quad m_p^2 = E \{ \epsilon_p^2 \} = C(y_p, y_p) - 2 \sum_{i=1}^n h_{p1} C(y_p, x_i) + \\ + \sum_{i=1}^n \sum_{j=1}^n h_{p1} h_{pj} C(x_i, x_j)$$

where

$$C(y_p, y_p) = E \{ \bar{y}_p^2 \} \quad \text{variance of } y_p$$

$$C(y_p, x_i) = E \{ y_p x_i \} \quad \text{cross-covariance of } y_p \text{ and } x_i$$

$$C(x_i, x_j) = E \{ x_i x_j \} \quad \text{auto-covariance of observations}$$

The minimum of m_p^2 is obtained by differentiation of (2.3) with respect to each of the weights h_{p1} , which gives the (generalized) Wiener-Hopf equations in the discrete case:

$$(2.4) \quad \sum_{i=1}^n h_{p1} C(x_i, x_j) = C(y_p, x_j); \quad j = 1, 2, \dots, n$$

or with obvious matrix notations

$$(2.4') \quad h C_{xx} = c_{yx}$$

with the solution

$$(2.5) \quad h = c_{yx} C_{xx}^{-1}$$

By inserting the solution for h into (2.3), the least squares prediction variance becomes (with matrix notations):

$$(2.6) \quad m_p^2 = C(y_p, y_p) - c_{yx} C_{xx}^{-1} c_{yx}^T$$

Let us now study the corresponding continuous problem. Suppose that the stochastic process x is recorded continuously along a profile, and that the process is stationary (such an assumption is normally imposed also in the practical application with discrete data as outlined above). Formula (2.1) is then defined:

$$(2.7) \quad \hat{y}(t) = \int_{-\infty}^{\infty} h(t-t') x(t') dt' = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

and the prediction variance (2.3) becomes

$$(2.8) \quad m^2 = \sigma_y^2 - 2 \int_{-\infty}^{\infty} h(\alpha) C_{yx}(\alpha) d\alpha + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) C_{xx}(\alpha - \beta) d\alpha d\beta$$

where σ_y^2 is the variance of y . In this case the least squares prediction problem is solved by the following Wiener-Hopf integral equation (Papoulis, 1965, p. 345):

$$(2.9) \quad \int_{-\infty}^{\infty} h(\alpha) C_{xx}(\tau - \alpha) d\alpha = C_{yx}(\tau)$$

The weight function $h(\alpha)$ is most conveniently obtained from the Fourier transform of (2.9). Let us use the notations:

$$H(\omega) = \tilde{h}(\tau), \quad S_{xx}(\omega) = \tilde{C}_{xx}(\tau), \quad S_{yx}(\omega) = \tilde{C}_{yx}(\tau)$$

where \tilde{X} denotes the Fourier transform of X , i.e.:

$$(2.10) \quad \tilde{X}(\omega) = \int_{-\infty}^{\infty} X(\tau) e^{-i\omega\tau} d\tau$$

Using the convolution theorem (Papoulis, *ibid.*, p. 159) the Fourier transform of (2.9) becomes:

$$H(\omega) S_{xx}(\omega) = S_{yx}(\omega)$$

Thus

$$(2.11) \quad H(\omega) = S_{yx}(\omega) / S_{xx}(\omega)$$

Finally, the weight function $h(\tau)$ is given by the inverse Fourier transform of $H(\omega)$:

$$(2.12) \quad h(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega\tau} d\omega$$

The prediction variance along the profile may be determined directly from the spectrum of (2.8), which becomes:

$$(2.13) \quad m^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{yy}(\omega) - H(\omega) S_{yx}(\omega)] d\omega$$

The basic formulae for this study are (2.7), (2.11), (2.12) and (2.13). In the numerical applications, the integrals are approximated by summations and the integration limits are truncated at finite values.

3. Prediction of Anomalies Along Arcs

In this section we are going to present the formulae to be used in a practical application of the Fourier transforms for the estimation of free-air point gravity anomalies along the arc from an (adjusted) altimeter data profile. The corresponding formulae for least squares filtering are given in Moritz (1967). By using a low degree spherical harmonic expansion as the reference field, the remaining altimeter residuals can be regarded as regional and local variations with mean value 0 {cf. (2.2)}. The purpose of the method is to improve the gravity anomalies provided by the spherical harmonic expansion by adding the information obtained from altimeter data. Rigorously, the method requires that the data have equal spacing. This will be used as an approximation for each arc.

The point gravity anomaly Δg_k is predicted from the residual altimeter height (geoidal undulation) $X = N - \bar{N}$ by {cf. (2.1)}:

$$(3.1) \quad \Delta \hat{g}_k = \Delta \bar{g}_k + \Delta t \sum_{j=0}^{n+1} h_j (X_{k-j} + X_{k+j})$$

where

$$\begin{aligned}\bar{N} &= \text{reference geoid undulation (in meters)} \\ \Delta\bar{g}_n &= \text{reference gravity anomaly (in mgals)} \\ \Delta t &= \text{spacing of data (in meters)} \\ m &= \text{selected number of discrete intervals of spectra}\end{aligned}$$

Δt is determined from the spherical distance ψ of the total arc segment and the number (n) of observations:

$$(3.2) \quad \Delta t = r_n \psi / (n - 1)$$

where r_n is the mean earth radius in the area. Furthermore, the fundamental wavelength:

$$(3.3) \quad \Delta\omega = \pi / m \Delta t$$

is computed. The choice of m has to be based on the required resolution of the solution. It remains to determine the weights h_j , which is done in the following way.

Covariance functions. In Moritz (ibid.) it is suggested to compute the covariance functions (C_{NN} , $C_{\Delta\epsilon N}$ and $C_{\Delta\epsilon \Delta\epsilon}$) from observations. However, as the only kind of observations available in the present case is altimeter data, we will use the empirical covariance functions given by the subroutine COVA of Tscherning and Rapp (1974). The low degree harmonics included in the reference field are deleted in the covariance functions. Computer time is gained by storing the covariance functions in a discrete table and then determining the necessary values by interpolation between table values. The covariance functions are computed for:

$$C_{NN}(\ell \Delta t), C_{\Delta\epsilon N}(\ell \Delta t) \text{ and } C_{\Delta\epsilon \Delta\epsilon}(\ell \Delta t); \ell = 0, 1, \dots, m$$

The observation errors are taken into account in the autocovariance matrix C_{NN} , which is replaced by:

$$(3.4) \quad \bar{C}_{NN}(\ell \Delta t) = \begin{cases} C_{NN}(0) + \sigma_N^2 & \text{if } \ell = 0 \\ C_{NN}(\ell \Delta t) & \text{if } \ell \neq 0 \end{cases}$$

where σ_N^2 is the mean square error of the observations along the arc:

$$\sigma_N^2 = \sum_{i=1}^n \frac{\sigma_i^2}{n}$$

σ_i being the standard error of the i -th observation.

Spectra. The following Fourier transformations are performed:

$$\bar{C}_{NN} \rightarrow \bar{S}_{NN}, \quad C_{\Delta \xi N} \rightarrow S_{\Delta \xi N} \quad \text{and} \quad C_{\Delta \xi \Delta \xi} \rightarrow S_{\Delta \xi \Delta \xi}$$

The numerical integration is as follows [cf. formula (2.10)]:

Raw values:

$$(3.5) \quad S_r' = \Delta t \left[C_0 + 2 \sum_{q=1}^{n-1} C_q \cos(qr\pi/m) + C_n \cos(r\pi) \right]$$

where

$$r = 0, 1, 2, \dots, m$$

Smoothed values:

$$(3.6) \quad \begin{aligned} S_0 &= \frac{1}{2} (S_0' + S_1') \\ S_q &= \frac{1}{4} (S_{q-1}' + 2S_q' + S_{q+1}') ; \quad q = 1, 2, \dots, m-1 \\ S_m &= \frac{1}{2} (S_{m-1}' + S_m') \end{aligned}$$

System function.

$$(3.7) \quad H_r = S_{\Delta \xi N}(r \Delta \omega) / \bar{S}_{NN}(r \Delta \omega) ; \quad r = 0, 1, 2, \dots, m$$

Weight function. Inverse Fourier transform [See (2.12)]:

$$H(\omega) \rightarrow h(\tau)$$

Raw values:

$$(3.8) \quad h'_\ell = \frac{\Delta\omega}{2\pi} \left[H_0 + 2 \sum_{q=1}^{m-1} H_q \cos(q\ell\pi/m) + H_m \cos(\ell\pi) \right]$$

where

$$\ell = 0, 1, 2, \dots, m$$

Smoothed values:

$$(3.9) \quad \begin{aligned} h_0 &= \frac{1}{4} (h'_0 + h'_1) \\ h_q &= \frac{1}{4} (h'_{q-1} + h'_q + h'_{q+1}) \quad ; \quad q = 1, 2, \dots, m-1 \\ h_m &= \frac{1}{4} (h'_{m-1} + 2h'_m) \\ h_{m+1} &= \frac{1}{4} h'_m \end{aligned}$$

In this way the necessary weights of (3.1) are determined.

Remark. In Moritz (Ibid., p. 49) the factor 1/4 of h_0 is incorrectly set to 1/2. For further details see the Appendix.

4. Prediction of Mean Anomalies

In Moritz (1969) a method is given for a least squares interpolation of data between parallel profiles. As the altimeter arcs are usually not parallel, this method cannot be used. However, the technique of predicting in points that are not on the profile of observations may be used for one arc at a time (extrapolation). This method requires that the shortest distance (a) from the profile to the point of computation is computed. Then the cross-covariance matrix used in the previous section is simply modified to:

$$(4.1) \quad C_{\Delta \xi N}(\tau) = G_{\Delta \xi N}(\alpha), \quad \alpha = \sqrt{a^2 + \tau^2}$$

where G is the output from subroutine COVA. The auto-covariance matrices C_{NN} and $C_{\Delta \xi \Delta \xi}$ are not modified. Thus the auto-spectra S_{NN} and $S_{\Delta \xi \Delta \xi}$ can be computed once and for all for each profile (arc). The cross-spectrum $S_{\Delta \xi N}$ and the weight function $h(\tau)$ must be recalculated for each new distance a . However, for a constant a the same weight function is used for repeated predictions. This fact allows for the computation of many points inside each block without significant increase of the cost. A further simplification of the mean block predictions is obtained by calculating $C_{\Delta \xi N}(\tau)$ as the average of three different a -values for each block:

$$(4.2) \quad C_{\Delta \xi N}(\tau) = \sum_{j=-1}^1 G \left(\sqrt{\tau^2 + (a + jy)^2} \right) / 3$$

where a is the shortest distance between the block center and the arc, and y is approximately half the side of the block. y is selected a priori by the user. If the profile was parallel to one side of the block, y could be selected half of the perpendicular side without introducing any error. However, in practice, the profiles are generally not parallel to any side of the block and for any choice of y an approximation error is present.

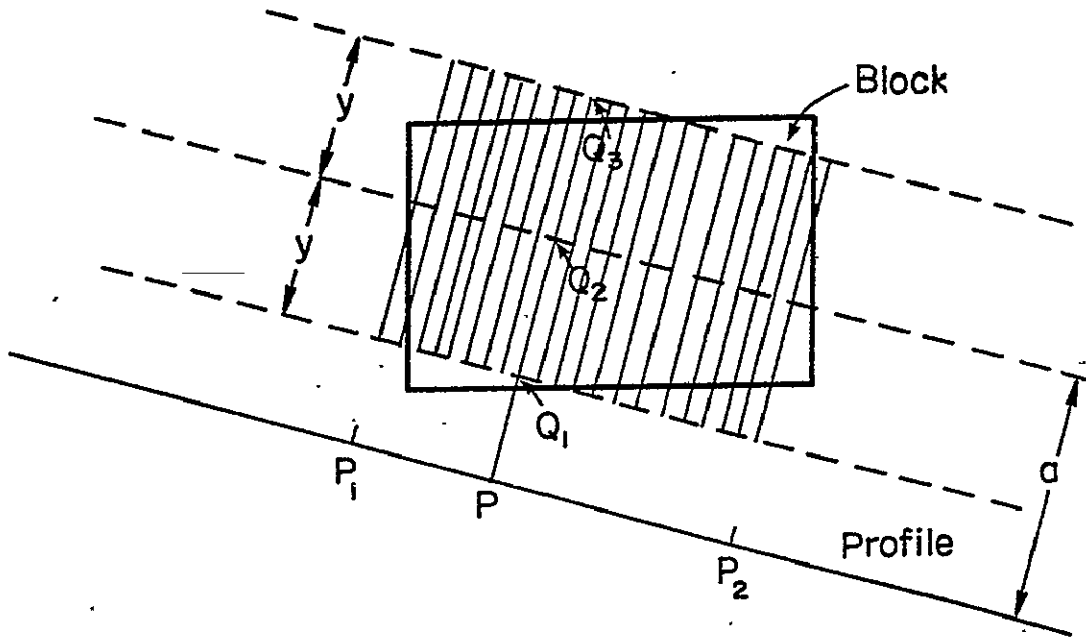


Figure 4.1. The cross-covariance function is applied as the average of the covariance functions among the point P on the profile and the points $Q_1 - Q_3$ on the normal to the profile. y is approximately half the side of the block. The predictions are performed for all points P with $P_1 \leq P \leq P_2$, and the block mean prediction is represented by the mean value of all individual predictions (in the shaded area).

5. Accuracy Estimation

The variance of prediction of y along a profile from the observations x was given in formula (2.12). In the practical application, this formula takes the form:

$$(5.1) \quad m^2 = \frac{\Delta\omega}{2\pi} \left(E_0 + 2 \sum_{q=1}^{m-1} E_q + E_m \right)$$

where $\Delta\omega$ was defined in (3.3) and

$$E_q = S_{yy}(q\Delta\omega) - H_q S_{yx}(q\Delta\omega); \quad q = 0, 1, 2, \dots, m$$

Now we proceed to estimate the errors of mean gravity anomaly predictions. The error covariance (after prediction) between the points P_1 and P_j are given by (Moritz, 1969, p. 41):

$$(5.2) \quad \sigma_{1j}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{1j}(\omega) e^{i\omega\tau} d\omega$$

where

$$E_{1j}(\omega) = S_{yy}(\omega) - H(\omega) S_{yx}(\omega)$$

$$\tau = \text{distance between } P_1 \text{ and } P_j$$

We define the mean value of the prediction $\hat{y}(t)$ over the interval $t_1 \leq t \leq t_2$ by:

$$(5.3) \quad \bar{y} = \frac{1}{2b} \int_{t_1}^{t_2} \hat{y}(t) dt$$

where

$$b = (t_2 - t_1)/2$$

The variance of \bar{y} is then given by:

$$(5.4) \quad \overline{m}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{11}(\omega) F(\omega) d\omega$$

where

$$F(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ [\sin(b\omega)/b\omega]^2 & \text{if } \omega \neq 0 \end{cases}$$

Proof: Let us denote the error of $\hat{y}(t)$ by $\epsilon(t)$. The error of the mean (\bar{y}) is then given by:

$$\bar{\epsilon} = \frac{1}{2b} \int_{t_1}^{t_2} \epsilon(t) dt$$

and

$$\begin{aligned} \overline{m}^2 &= E\{\bar{\epsilon}^2\} = \frac{1}{4b^2} E\left\{ \int_{t_1}^{t_2} \int_{t_1}^{t_2} \epsilon(t) \epsilon(t') dt dt' \right\} = \\ &= \frac{1}{4b^2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sigma_{11}(t-t') dt dt' \end{aligned}$$

Inserting (5.2) we obtain:

$$\overline{m}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{11}(\omega) F(\omega) d\omega$$

where

$$F(\omega) = \frac{1}{4b^2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} e^{i\omega(t-t')} dt dt' = \begin{cases} 1 & \text{for } \omega = 0 \\ (\sin(b\omega)/b\omega)^2 & \text{for } \omega \neq 0 \end{cases}$$

Q.E.D.

Using the notation

$$J_r = E_{1j}(\omega_r) F(\omega_r)$$

We finally arrive at the following practical formula:

$$(5.5) \quad \overline{m}^2 = \frac{\Delta\omega}{2\pi} \left(J_0 + 2 \sum_{q=1}^{n-1} J_q + J_n \right)$$

As the cross-covariances that are used in the prediction of mean gravity anomalies are assumed to be mean values across each block perpendicular to the altimeter profile (cf. section 4), formula (5.5) may be applied directly for estimating the prediction error of the block mean value. In this case, the constant $2b$ is the side of the block.

Special cases:

a) $b \rightarrow 0$ implies $F(\omega) \rightarrow 1$ for all ω and therefore $\overline{m}^2 \rightarrow m^2$, i.e. we arrive at the case of point estimation.

b) $b \rightarrow \infty$ implies $F(\omega) \rightarrow \delta(\omega)$ and $\overline{m}^2 \rightarrow E_{1j}(0)/2\pi$.

6. Computations

In all computations equidistant spacing of the data along each arc is assumed, and the mean earth radius (r_m) is set to 6370 km. First, the Fourier transform method described in section 3 was tested for least squares filtering of the data along some arcs. The filter is given by formula (3.1) with Δg_k and $\overline{\Delta g}_k$ substituted by \overline{N}_k and \overline{N}_k and $S_{\Delta g N}$ of (3.7) substituted by S_{NN} . The GEM 7 spherical harmonic expansion to degree 16 was used as a reference. An example of this test is shown in Figure 6.1. A reasonable smoothing of the raw altimeter data is obtained.

Second, mean gravity anomalies were predicted from altimeter data. The spherical harmonic coefficients of GEM 9 to degree 20 were used in the reference field. The covariance output from subroutine COVA was stored in tables at 0.1 intervals. Each profile was processed separately. The final mean anomaly of a block ($\overline{\Delta g}$) was estimated by the weighted average of the outcomes ($\overline{\Delta g}_i$) from several arcs in the area:

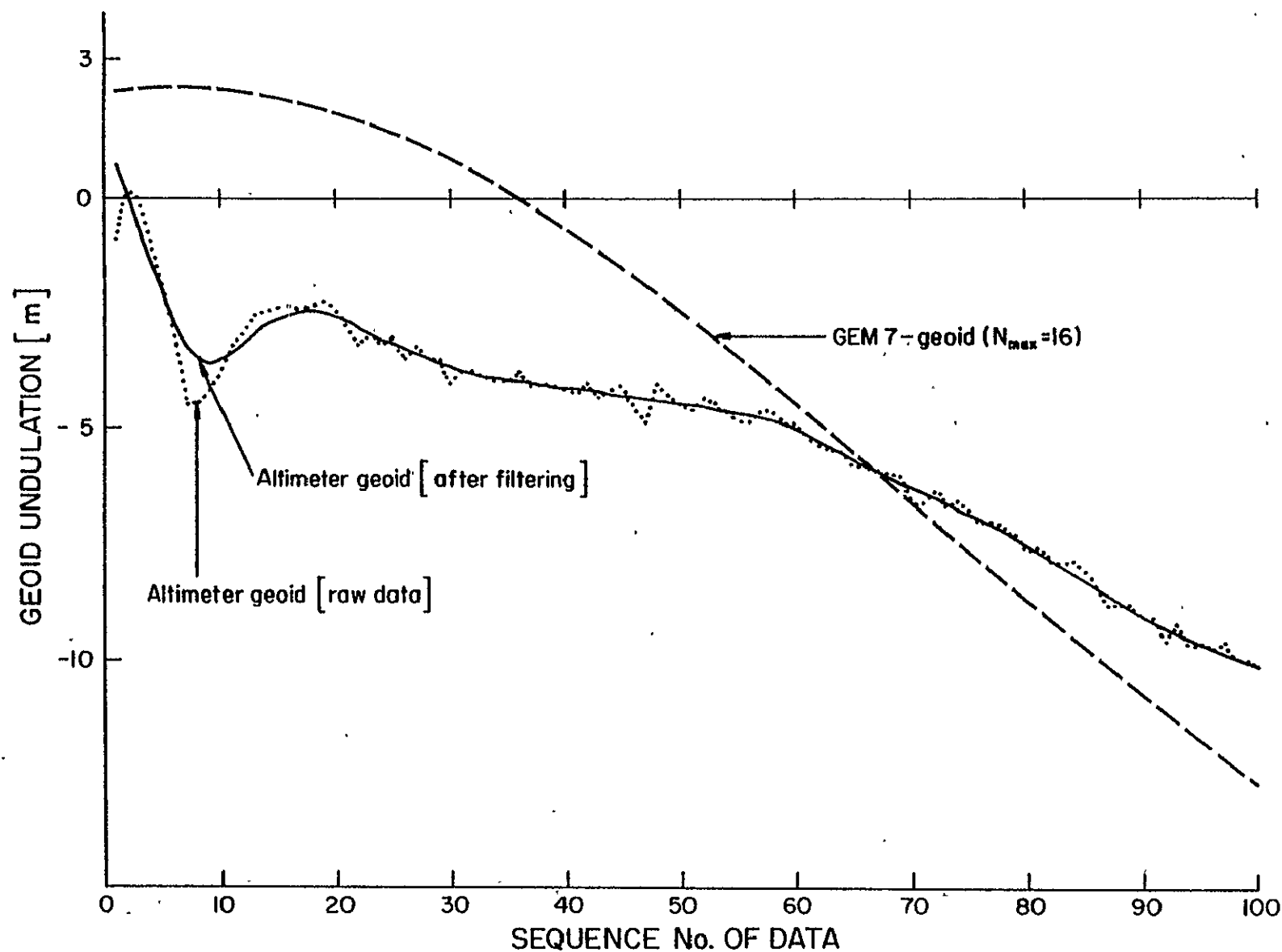


Figure 6.1. Filtering of 300 altimeter observations along arc 2859 using Fourier transforms. The RMS standard deviation of the observations is 0.68 m. Number of discrete intervals of the spectra: 75.

$$(6.1) \quad \Delta \bar{g} = \bar{m}^2 \sum_{i=1}^n \Delta \bar{g}_i / m_i^2$$

where \bar{m}^2 is the estimated variance of the mean:

$$(6.2) \quad \bar{m}^2 = 1 / \sum_{i=1}^n \bar{m}_i^{-2}$$

and m_i^2 are the estimated variances of predictions $\Delta \bar{g}_i$.

Requirements of used arcs. Each segment of an arc with less than 100 sequential observations [with gaps less than 0.5° (0.3°) for 5° ($1^\circ \times 1^\circ$) anomalies] were rejected. The maximum accepted distance from an arc to the center of a block was 3° (if not specially mentioned). For details of the adjusted altimeter data, we refer to Rapp (1977 b).

The cross-covariance matrix $C_{\Delta \bar{g} N}$ was computed as the average of 3 points perpendicular to the profile [see formula (4.2)]. The 5° ($1^\circ \times 1^\circ$) mean anomalies were estimated as the averages of 21 (5) such sequential "mean-value points" predictions inside each block parallel to the profile. The area thus covered with point predictions is illustrated as the shaded area of Figure 4.1. Again we emphasize that for any choice of y and number of "mean-value points" inside a block for forming the block average, an approximation error occurs due to the inclusion of points outside the block and (or) the exclusion of points inside the block (Figure 4.1). However, this type of error is more or less present in any numerical integration and as long as the error does not become critical, the quality of the final prediction should be compared with the necessary computer time. In this respect, the selected method is favorable.

The number of discrete intervals (m) in the numerical integrations of the spectra was set to 60 and 75 for 5° and $1^\circ \times 1^\circ$ anomaly predictions, respectively. The prediction results are compared with the terrestrial mean anomalies described in Rapp (1977a).

6.1 The Calibration Area

Data: Adjusted altimeter data in the area $10^\circ < \varphi < 50^\circ$ and $275^\circ < \lambda < 310^\circ$ (see Rapp, 1977b).

In this area 6 5° equal area anomalies and 30 $1^\circ \times 1^\circ$ anomalies were estimated (all the $1^\circ \times 1^\circ$ blocks are inside 5° block No. 403; see Rapp, 1977a). The prediction results of the 5° blocks are shown in Tables 6.1 and 6.2. These compu-

tations include approximately 12800 observations in 85 arc segments, and the total computer time (IBM 370/168) was 2.1 minutes. The RMS differences prediction - terrestrial and GEM 9 - terrestrial anomaly are 3.7 and 7.1 mgal, respectively.

The prediction results of the 30 1° x 1° anomalies are given in Tables 6.3 - 6.4. Approximately 6550 observations in 41 arc segments were processed, and the total computer time was 8 minutes. The RMS differences as above are 11.5 and 13.8 mgal. However, as the RMS standard error of the terrestrial data is as large as 13.6 mgal, these differences might well be due to the uncertainty of the terrestrial gravity information alone. Subsequently, by excluding the 9 last blocks (with poor terrestrial information) from the comparison, the remaining subset of 21 blocks gives the following RMS differences:

$$|\Delta g_A - \Delta g_T| = 4.8 \text{ mgal} \quad \text{and} \quad |\Delta g_{\text{GEM}} - \Delta g_T| = 6.7 \text{ mgal}$$

Table 6.1

5° Equal Area Predictions in the Calibration Area
Block Numbers and Terrestrial Anomalies ($\Delta g_T \pm m_T$) Refer to Rapp (1977a).
Units: mgal

Block Number Coordinates		Predic. Means		m_A^*	Spread [†]	No. of Arcs (n)	$\Delta g_T \pm m_T$	GEM 9
		Wtd.	Direct					
342	40°-35°, 284°-291°	-23.7	-24.7	1.2	3.8	41	-21.9 ± 1.7	-17.0
343	40°-35°, 291°-297°	-20.0	-20.1	1.0	3.5	37	-24.3 ± 2.0	-17.5
402	35°-30°, 283°-289°	-28.0	-29.9	0.9	4.7	44	-34.1 ± 2.4	-20.6
403	35°-30°, 289°-295°	-24.5	-23.1	1.0	5.9	44	-26.8 ± 2.5	-27.7
465	30°-25°, 281°-287°	-18.2	-17.3	0.9	2.9	41	-15.4 ± 2.8	-19.2
466	30°-25°, 287°-293°	-33.4	-31.2	1.0	4.1	34	-29.8 ± 2.8	-35.9

* estimated prediction error from altimeter data

[†] = $[\sum (\Delta g_i - \bar{\Delta g})^2 / n]^{1/2}$, where Δg_i = prediction from arc i.

ORIGINAL PAGE IS
OF POOR QUALITY

Table 6.2

Differences Between Predicted, Terrestrial and GEM 9 Anomalies
in the Calibration Area
Units: mgal

Block No.	Weighted Terr.	Direct Terr.	GEM 9 Terr.	Weighted GEM 9
342	-1.8	-2.8	4.9	-6.7
343	4.3	4.2	6.8	-2.5
402	6.1	4.2	13.5	-7.4
403	2.3	3.7	-0.9	3.2
465	-2.8	-1.9	-3.8	1.0
466	-3.6	-1.4	-6.1	2.5
RMS:	3.8	3.2	7.1	4.5

Table 6.3

30 1° x 1° Mean Anomaly Predictions from Altimeter Data
in the GEOS-3 Calibration Area.

Δg_T and m_T According to Rapp (1977a)

Units: mgal

$\varphi^\circ, \lambda^\circ$ (NW corner)		Predic. Means		\bar{m}_A^*	Spread [†]	# of Arcs	$\Delta g_T \pm m_T$		GEM 9
		Wtd.	Direct						
35	289	-28.0	-28.0	3.1	6.3	19	-33	6	-22.9
35	290	-27.2	-26.3	3.0	5.3	20	-24	7	-23.6
35	291	-25.2	-24.4	3.0	4.6	19	-29	2	-23.8
35	292	-22.7	-21.1	2.8	5.1	25	-30	2	-23.6
35	293	-19.3	-17.3	2.7	5.8	29	-24	3	-23.1
35	294	-14.4	-13.5	3.2	6.4	26	-17	11	-22.2
34	289	-29.6	-30.6	3.5	7.6	18	-32	14	-25.1
34	290	-26.1	-26.9	3.4	6.6	19	-31	11	-25.8
34	291	-24.1	-23.4	2.9	5.9	24	-26	15	-26.1
34	292	-21.7	-19.7	2.8	6.6	27	-25	10	-25.8
34	293	-17.9	-14.8	2.9	9.3	27	-25	11	-25.2
34	294	-11.9	-9.6	3.2	10.9	25	-16	15	-24.2
33	289	-32.8	-32.9	3.4	7.1	20	-38	16	-27.4
33	290	-28.6	-28.5	3.2	6.5	22	-32	18	-28.2
33	291	-24.6	-24.2	3.2	6.7	24	-28	12	-28.4
33	292	-20.6	-19.7	3.2	7.6	26	-30	14	-28.2
33	293	-15.5	-13.4	3.2	10.4	24	-25	14	-27.4
33	294	-9.4	-5.6	3.2	19.8	25	-9	6	-26.3
32	289	-36.8	-36.0	3.3	6.7	19	-40	19	-29.6
32	290	-32.3	-31.0	3.2	7.1	24	-35	19	-30.5
32	291	-27.2	-26.2	3.1	7.5	27	-26	13	-30.8
32	292	-22.3	-21.2	3.2	8.5	26	-41	22	-30.5
32	293	-18.8	-17.3	3.3	8.8	23	-23	17	-29.8
32	294	-12.2	-12.4	3.7	8.7	21	-15	14	-28.5
31	289	-36.1	-35.2	3.0	7.0	23	-11	15	-31.8
31	290	-33.3	-31.6	3.2	7.6	26	4	15	-32.7
31	291	-29.1	-26.3	3.3	8.8	25	-2	15	-33.1
31	292	-23.3	-21.5	3.2	10.5	25	-6	15	-32.9
31	293	-17.1	-16.3	3.6	9.8	21	-10	15	-32.0
31	294	-13.7	-14.6	3.9	8.4	19	-8	14	-30.7

* * estimated prediction error from altimeter data

[†] = $[\Sigma(\Delta g_i - \bar{\Delta g})^2 / n]^{1/2}$, where Δg_i = prediction from arc i

Table 6.4

Differences Between Altimeter (Weighted and Direct Means),
Terrestrial and GEM 9 Anomalies According to Table 6.3

Units: mgal

$\varphi^\circ, \lambda^\circ$ (NW corner)	W - Terr.	D - Terr.	GEM Terr.	W - GEM
35, 289	5.0	5.0	10.1	-5.1
35, 290	- 3.2	- 2.3	0.4	-3.6
35, 291	3.8	4.6	5.2	-1.4
35, 292	7.3	8.9	6.4	0.9
35, 293	4.7	6.7	0.9	3.8
35, 294	2.6	3.5	- 5.2	7.8
34, 289	2.4	1.4	6.9	-4.5
34, 290	4.9	4.1	5.2	-0.3
34, 291	1.9	2.6	- 0.1	2.0
34, 292	3.3	5.3	- 0.8	4.1
34, 293	7.1	10.2	- 0.2	7.3
34, 294	4.1	6.4	- 8.2	12.3
33, 289	5.2	5.1	10.6	-5.4
33, 290	3.4	3.5	3.8	-0.4
33, 291	3.4	3.8	- 0.4	3.8
33, 292	9.4	10.3	1.8	7.6
33, 293	9.5	11.6	- 2.4	11.9
33, 294	- 0.4	3.4	-17.3	16.9
32, 289	3.2	4.0	10.4	-7.2
32, 290	2.7	4.0	4.5	-1.8
32, 291	- 1.2	- 0.2	- 4.8	3.6
32, 292	18.7	19.8	10.5	8.2
32, 293	4.2	5.7	-6.8	11.0
32, 294	2.8	2.6	-13.5	16.3
31, 289	-25.1	-24.2	-20.8	-4.3
31, 290	-37.3	-35.6	-36.7	-0.6
31, 291	-26.1	-24.3	-31.1	5.0
31, 292	-17.3	-15.5	-26.9	9.6
31, 293	- 7.1	- 6.3	-22.0	14.9
31, 294	- 5.7	- 6.6	-22.7	17.0
RMS	11.5	11.4	13.8	8.3
RMS from first 21 blocks	4.8	5.9	6.7	6.8

6.2 The Philippines Area

Data: Adjusted altimeter data in the area $0^\circ < \varphi < 40^\circ$ and $120^\circ < \lambda < 160^\circ$. (For details see Rapp, 1977b.)

In this area 16 5° mean anomalies and 5 $1^\circ \times 1^\circ$ anomalies were predicted. For the prediction of the 5° anomalies approximately 11300 observations in 48 arc segments were included and the total computation time (IBM 370/168) was 1.5 minutes. The results are shown in Tables 6.5 and 6.6. The RMS differences between predicted (weighted) and terrestrial (see Rapp, 1977a) anomalies was 7.7 mgal. The corresponding difference between the GEM 9 and terrestrial anomalies was 11.2 mgal.

The 5 $1^\circ \times 1^\circ$ blocks were particularly selected across a trench with very large variations of the terrestrial anomalies. The results of these predictions are given in Table 6.7. Large discrepancies are obtained between predicted and terrestrial anomalies. The main reason for the bad results is probably the following. Theoretically, the method requires that the profiles of observations are extended to infinity [formula (2.7)]. In practice, we can usually truncate the arcs at, say, 5° outside the prediction area without significant loss of information. However, in the present case the land masses (islands) on the west side of the trench truncate the arcs at a few degrees, which has a serious impact on the prediction results in such a disturbed area.

For comparison, we report some $1^\circ \times 1^\circ$ anomaly predictions across the Mariana Trench, where there are no disturbing land masses (Table 6.8). An obvious improvement of the GEM 9 anomalies is obtained, and the result is rather good if we consider the limited number of arcs. The maximum allowed distance arc-center of block seems to be very important in these very disturbed areas. See section 6.3.

6.3 Dependence on the Number of Arcs

In the previous computations, the maximum perpendicular distance from the center of a block to an altimeter profile was 3° . It is of interest to find out whether a change of this distance of truncation (not to be mixed with the length of the altimeter profile outside the prediction area) will change the prediction results significantly. The RMS differences between predicted mean anomalies and terrestrial anomalies for various such distances are given in Table 6.9. In general, the differences are decreasing with increasing truncation distance, at least to $3^\circ.5$. However, in the Mariana Trench the result is opposite. We conclude that a smaller distance of truncation should be adopted in regions with large disturbances than in gentle areas.

Table 6.5

5° Equal Area Predictions in the Philippines Area
 Block Numbers and Terrestrial Anomalies ($\Delta g_T \pm m_T$) Refer to Rapp (1977a)
 Units: mgal

Block No.	Coordinates	Pred. Means		\bar{m}_A^*	Spread ⁺	# of Arcs (n)	$\Delta g_T \pm m_T$	GEM 9
		Wtd.	Direct					
438	30°-25°, 129°-135°	-0.6	-0.1	3.4	3.2	6	-8.4 ± 2.8	11.9
439	30°-25°, 135°-141°	20.0	19.9	4.4	4.6	7	20.9 ± 2.8	13.4
440	30°-25°, 141°-146°	3.1	6.5	4.5	8.0	7	20.1 ± 2.5	10.5
441	30°-25°, 146°-152°	-0.3	1.2	5.9	2.5	3	0.6 ± 2.8	5.9
503	25°-20°, 129°-134°	6.5	5.9	2.0	1.9	10	7.8 ± 2.7	2.5
504	25°-20°, 134°-140°	14.2	14.8	2.5	3.5	17	10.6 ± 2.7	10.4
505	25°-20°, 140°-145°	19.3	18.3	3.1	10.5	11	20.1 ± 2.9	13.7
506	25°-20°, 145°-150°	3.4	4.1	3.8	8.5	10	-0.7 ± 2.9	9.6
571	20°-15°, 130°-136°	5.3	5.4	1.6	1.6	17	1.6 ± 2.8	4.9
572	20°-15°, 136°-141°	13.4	13.4	2.6	2.5	14	10.0 ± 2.9	7.8
573	20°-15°, 141°-146°	20.0	20.4	3.5	5.3	12	35.2 ± 3.0	9.5
574	20°-15°, 146°-151°	6.2	1.7	3.1	13.0	11	-5.3 ± 3.5	5.2
640	15°-10°, 129°-134°	11.7	11.7	1.8	2.1	16	12.4 ± 3.1	15.3
641	15°-10°, 134°-139°	12.0	12.1	1.6	2.9	15	10.3 ± 3.2	10.6
642	15°-10°, 139°-144°	4.8	4.5	3.0	11.5	13	-8.7 ± 3.3	7.7
643	15°-10°, 144°-149°	-8.2	-12.9	3.1	19.7	11	-8.3 ± 2.8	5.6

* estimated prediction error from altimeter data

⁺ = $[\sum(\Delta g_i - \bar{\Delta g})^2 / n]^{1/2}$, where Δg_i = prediction from arc i

Table 6.6

Differences Between Predicted, Terrestrial and GEM 9 Anomalies
in the Philippines Area

Units: mgal

Block No.	Weighted Terr.	Direct Terr.	GEM 9 Terr.	Weighted GEM 9
438	7.8	8.3	20.3	-12.5
439	- 0.9	- 1.0	- 7.5	6.6
440	-17.0	-13.6	- 9.6	- 7.5
441	- 0.9	0.6	5.3	- 6.2
503	- 1.3	- 1.9	- 5.3	4.1
504	3.6	4.2	- 0.2	3.7
505	- 0.8	- 1.8	- 6.4	5.6
506	4.1	4.8	10.3	- 6.3
571	3.7	3.8	3.3	0.4
572	3.4	3.4	-2.2	5.6
573	-15.2	-14.8	-25.7	10.5
574	11.5	7.0	10.5	1.0
640	- 0.7	- 0.7	2.9	- 3.6
641	1.7	1.8	0.3	1.4
642	13.5	13.2	16.4	- 2.9
643	0.1	- 4.6	13.9	-13.8
RMS:	7.7	7.1	11.2	6.9

Table 6.7

Prediction of $1^\circ \times 1^\circ$ Anomalies in the Philippine Trench.
 Terrestrial Anomalies (Δg_T) According to Rapp (1977a). Units: mgal.
 Maximum Allowed Distance Arc-Center of Block: 3° .

ϕ NW Corner	Pred. Means		m_A^*	Spread [†]	# of Arcs (n)	Δg_T $\pm m_T$	GEM 9
	Wtd.	Direct					
$11^\circ, 124^\circ$	22	19	13	8	5	70 ± 12	33
$11^\circ, 125^\circ$	31	21	8	38	9	153 ± 20	31
$11^\circ, 126^\circ$	15	8	7	28	11	-107 ± 25	29
$11^\circ, 127^\circ$	20	14	6	22	12	31 ± 10	27
$11^\circ, 128^\circ$	21	18	5	14	14	43 ± 10	25

* estimated prediction error from altimeter data

$\dagger = [\Sigma(\Delta g_i - \Delta \bar{g})^2 / n]^{1/2}$, where Δg_i is the prediction from arc i.

Table 6.8

Prediction of $1^\circ \times 1^\circ$ Anomalies in the Mariana Trench.
 Terrestrial Anomalies According to Watts (1975). Units: mgal.
 Maximum Distance Arc-Center of Block: 2° .

ϕ NW Corner	Pred. Means		m_A^*	Spread [†]	# of Arcs (n)	Δg_T $\pm m_T$	GEM 9
	Wtd.	Direct					
$12^\circ, 139^\circ$	6	5	16	10	3	8 ± 15	9
$12^\circ, 140^\circ$	10	10	30	-	1	-9 ± 20	8
$12^\circ, 141^\circ$	- 12	- 13	23	11	2	-172 ± 25	8
$12^\circ, 142^\circ$	-111	-117	25	44	2	-172 ± 25	7
$12^\circ, 143^\circ$	- 65	- 61	18	75	4	-142 ± 25	6
$11^\circ, 139^\circ$	15	15	21	0	2	18 ± 15	10
$11^\circ, 140^\circ$	- 25	- 25	31	-	1	50 ± 15	9
$11^\circ, 141^\circ$	2	2	23	4	2	-72 ± 20	8
$11^\circ, 142^\circ$	4	4	24	6	2	-2 ± 15	7
$11^\circ, 143^\circ$	1	3	20	19	3	15 ± 10	7

* estimated prediction error from altimeter data

$\dagger = [\Sigma(\Delta g_i - \Delta \bar{g})^2 / n]^{1/2}$, where Δg_i is the prediction of arc i.

Table 6.9

RMS Differences (Weighted) Predicted - Terrestrial Mean Anomalies
for Various Distances of Truncation,
i.e. Maximum Distances Arc - Center of Block

Area Blocksize	No. of Blocks	Truncation Distance						
		1°	1°5	2°	2°5	3°	3°5	4°
Calibration 1° x 1°	30		13.3	12.2	11.7	11.5	11.5	11.7
Calibration 1° x 1°	21		7.8	6.2	5.2	4.8	4.7	4.6
Calibration 5°	6			4.1	4.0	3.8	3.8	4.1
Philippines 5°	16	10.0	7.6	7.7	7.9	7.7		
Mariana Trench 1° x 1°	10		69	68	75	80	81	87

Units: mgal

Finally, we compare 21 $1^\circ \times 1^\circ$ anomaly predictions for different numbers of included arcs (estimates). These numbers are selected simply by including only every 2 - 5 estimate of the original set of estimates. The results are given in Table 6.10.

Table 6.10

The Relation Between the Mean Number of Estimates (arcs)/Block and the RMS Difference Prediction-Terrestrial Anomaly ($1^\circ \times 1^\circ$).
Units: mgal.

Mean No. of Arcs/block	29.3	14.9	9.8	7.4	5.8
RMS Difference	4.6	4.9	4.8	5.5	6.7

The table shows a trend of decreasing differences with increasing number of arcs. We can therefore expect that the inclusion of all available arcs in the area will improve the prediction results of sections 6.1 and 6.2.

6.4 Test of Accuracy Estimates

The tables of sections 6.1 and 6.2 include the error estimates \overline{m}_A and m_T of the estimates from altimeter and terrestrial data, respectively. As these different set of data are uncorrelated, we have

$$\sigma_{TOT}^2 = E \{ (\Delta g_A - \Delta g_T)^2 \} = E \{ \overline{m}_A^2 \} + E \{ m_T^2 \}$$

or

$$(6.3) \quad \sigma_{TOT}^2 = \sigma_A^2 + \sigma_T^2$$

where σ_A^2 and σ_T^2 are the theoretical variances of altimeter and terrestrial data. Now we assume that the error of each Δg_A and Δg_T estimate is normally distributed and that the variances are approximately constant in each prediction area. Then (6.3) may be tested for the RMS estimates of σ_{TOT}^2 , σ_A^2 and σ_T^2 of each area, because $(\Delta g_A - \Delta g_T)^2 / (\overline{m}_A^2 + m_T^2)$ is approximately F - distributed with f - degrees of freedom (f being the number of blocks). This test is performed in Table 6.11, where F is the ratio of $(\Delta g_A - \Delta g_T)^2$ and $\overline{m}_A^2 + m_T^2$, with the largest outcome in the numerator. This ratio is compared with $F_{f, f, \alpha/2}$, where α is the risk level. Formula (6.3) is accepted if $F < F_{f, f, \alpha/2}$. The test shows

that the hypothesis (6.3) is accepted in the calibration area, but is rejected in the Philippines area. Although the assumptions of constant prediction error within each area and uncorrelated estimates of various blocks are approximations, the test gives some idea of the validity of the estimated prediction errors. We conclude that only in the Philippines area (6.3) is rejected, probably due to too small error estimates \overline{m}_A .

Table 6.11

Test of Formula (6.3)
Risk level (α) = 5%

Area Blocksize	# of blocks (f)	RMS (mgal)			F	$F_{r,f,\alpha/2}$	$\sigma_{TOT}^2 = \sigma_A^2 + \sigma_T^2 ?$
		Alt. Terr.	\overline{m}_A	m_T			
Calibration 1° x 1°	30	11.5	3.2	13.6	1.5	2.1	Yes
Calibration 5°	6	3.8	1.0	2.4	2.1	5.8	Yes
Philippine 5°	16	7.7	3.3	2.9	3.1	2.8	No

7. Conclusions and Final Remarks

In this report we have shown that if the data are given along profiles (arcs) an approximate version of least squares collocation may be performed in the frequency domain rather than in the direct domain. In this way the auto-covariance matrix, that has to be inverted, is transformed to diagonal form. In the present application, each profile is processed separately and the estimates from all arcs are finally joined in mean estimates. A considerable gain in computer time is obtained in comparison with conventional collocation technique. According to Thomas (1977) the necessary computer time for n observations is reduced from $\sim n^3$ to $\sim n \log n$ by using the Fourier transform method. For example, approximately 11300 observations in 48 arc segments were included in one of the present computations. 16 5° anomalies were predicted in 1.5 minutes computer time (IBM 370/168). In addition, new information that becomes available can easily be included in the predictions.

One should also consider that numerical difficulties occur in the direct method when the positions of two or more observations are close. Thus a filtering of the data is necessary, which probably implies a loss of information. The Fourier transform technique does not suffer from such limitations.

A disadvantage of the method is the requirement of equal spaced data along each arc. As the altimeter observations do not have a regular spacing along the profile (but do in time), such an approximation might be erroneous for point estimation, but will hardly effect the mean block predictions.

A summary of the computations is given in Table 7.1. The altimeter predictions are weighted means.

Table 7.1

Summary of Prediction Results
Approximate Numbers of Used Arcs and Observations
Computer: IMB 370/168

Area Blocks	No. of Obs.	No. of Arcs	No. of Blks	m	RMS Diff. (mgal)		\overline{m}_A^*	m_T^\dagger	CPU Time (min.)
					GEM Terr.	Pred. Terr.	(mgal)		
Philippine 5°	11300	48	16	60	10.5	7.7	3.3	2.9	1.5
Calibration 5°	12800	85	6	60	7.1	3.7	1.0	2.4	2.1
Calibration 1°	6550	41	30	75	13.8	11.5	3.2	13.6	8.0
Calibration 1°	6550	41	21	75	6.7	4.8	3.1	12.4	

* estimated prediction error from altimeter data

† estimated prediction error from terrestrial data

The table shows that a reasonable improvement of the GEM 9 anomalies is obtained by adding the altimeter information. However, this improvement is less obvious for the $30\ 1^\circ \times 1^\circ$ anomalies, because of the large errors of the terrestrial data. In section 6.3 we have demonstrated that the inclusion of more arcs in the predictions will probably give even better results. Due to the simplicity of the method, we can easily include all available arcs in the area. As in the direct collocation method, mean geoid undulations may be estimated by substituting the covariance function $C_{\Delta EN}$ by C_{NN} .

We feel that the estimated prediction errors are somewhat too optimistic. However, only for the $6\ 5^\circ$ equal area blocks in the Philippines area we found some evidence that this is the case.

Finally, it should be stated that the method fails in coast (or island) areas with large disturbances. The reason is probably the short truncation distances of the arcs in the direction of the land masses. Further investigations are needed to reveal the impact on the final predictions of the truncation distances (both with respect to the length of the arc and with respect to the distance arc - predicted block), the number of discrete intervals (m) in the numerical integrations and the choice of covariance functions.

References

- Blackman, R. B. and J. W. Tukey, The Measurement of Power Spectra from the Point of View of Communication Engineering, Dover Publications, Inc., New York, 1958.
- Moritz, H., Optimum Smoothing of Aerial Gravity Measurements, Department of Geodetic Science Report No. 81, The Ohio State University, Columbus, 1967.
- Moritz, H., A General Theory of Gravity Processing, Department of Geodetic Science Report No. 122, The Ohio State University, Columbus, 1969.
- Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Company, New York, 1965.
- Rapp, R. H., Potential Coefficient Determinations from 5° Terrestrial Gravity Data, Department of Geodetic Science Report No. 251, The Ohio State University, Columbus, 1977a.
- Rapp, R. H., Incomplete Final Results for Sea Surface Topography and Gravity Anomalies from Geos-3 Altimeter Data, Paper presented at the Geos-3 Conference, New Orleans, Louisiana, November, 1977b.
- Thomas, S. W., Fast Computational Methods for Least-Squares Gravity Data Processing, Paper presented at the AGU Spring Meeting, Washington D. C., 1977.
- Tscherning, C. C. and R. H. Rapp, Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree Variance Models, Department of Geodetic Science Report No. 208, The Ohio State University, Columbus, 1974.
- Watts, A., Gravity Field of the Northwest Pacific Ocean Basin and Its Margin: Philippine Sea, Geological Survey of America Map MC-12, 1975.

Appendix

We are going to show that the weight:

$$(A.1) \quad h_0 = \frac{1}{2} (h_0' + h_1')$$

according to Moritz (1967, p. 49) is incorrect and that the correct formula is:

$$(A.1') \quad h_0 = 1/4 (h_0' + h_1')$$

For the proof we need the following identity given by Blackman and Tukey (1958, p. 80):

$$\sum_{-a}^b \cos (\psi + 2hu) = \frac{\sin \{(a+b+1)u\}}{\sin u} \cos \{\psi + (b-a)u\}; \quad u \neq 0$$

Inserting $a = -1$, $b = m-1$, $\psi = 0$ and $u = \ell\pi/2m$, we arrive at the following formula:

$$(A.2) \quad \sum_{k=1}^{m-1} \cos (k\ell\pi/m) = \frac{\sin \{(m-1)\ell\pi/2m\}}{\sin (\ell\pi/2m)} \cos \frac{\ell\pi}{2} =$$

$$= \begin{cases} 0 & \text{for } \ell = 1, 3, 5, \dots \\ (-1)^{\ell/2} & \text{for } \ell = 2, 4, 6, \dots \end{cases}$$

Furthermore

$$(A.3) \quad \sum_{k=1}^{m-1} \cos (k\ell\pi/m) = m-1 \quad \text{for } \ell = 0$$

The relations (A.2) and (A.3) will be used in the following derivation of the weight h_0 .

From (3.8) or Moritz (ibid.) we have:

$$h_\ell' = \frac{\Delta\omega}{2\pi} \left[H_0 + 2 \sum_{q=1}^{m-1} H_q \cos (q\ell\pi/m) + H_m \cos (\ell\pi) \right]$$

In the special case of pure prediction (of X from the true values of X) all system functions H_ℓ are unity. Thus we obtain:

$$h'_\ell = c \left[1 + 2 \sum_{q=1}^{m-1} \cos (q\ell\pi/m) + \cos (\ell\pi) \right]$$

where {from formula (3.3)} :

$$c = \Delta\omega/2\pi = 1/2 m \Delta t$$

Inserting (A.2) and (A.3) we arrive at:

$$(A.4) \quad h'_\ell = \frac{1}{\Delta t} \begin{cases} 1 & , \quad \ell = 0 \\ 0 & , \quad \ell = 1, 3, 5, \dots \\ [1+(-1)^{\ell/2}]/m & , \quad \ell = 2, 4, 6, \dots \end{cases}$$

Next we determine the smoothed weights from the raw values {see formula (3.9) and Moritz (ibid.)}. However, we use the modification:

$$(A.5) \quad h_0 = p (h'_0 + h'_1)$$

where p is a constant to be determined in such a way that the prediction by formula (3.1) is correct (without error), i. e.

$$(A.6) \quad \hat{X}_n = \Delta t \sum_{j=0}^{m+1} h_j (X_{k-j} + X_{k+j}) \equiv X_k$$

For simplicity we assume that:

$$X_k = X = \text{constant for all } k$$

Furthermore, we assume that m is so large that (A.4) becomes (without loss of significance):

$$(A.7) \quad h'_\ell = \frac{1}{\Delta t} \begin{cases} 1 & \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the smoothed weights are given by (3.9) with the modification (A.5).
 Inserting (A.7) we obtain:

$$h_0 = p/\Delta t \quad , \quad h_1 = 1/4 \Delta t$$

and
$$h_q = 0 \quad \text{for} \quad q > 1$$

Combining these weights with (A.6), we obtain the following equation:

$$X(p + p + 1/4 + 1/4) \equiv X$$

with the solution for p:

$$p = 1/4$$

Thus we have proved that:

$$h_0 = 1/4 (h_0' + h_1')$$

is the correct expression for h_0 .

The Determination of Gravity Anomalies from Geoid Heights
Using Least Squares Collocation*

by

Richard H. Rapp

*paper originally presented at the 1976 Spring American
Geophysical Union Meeting, Washington, D. C., under
the title "Anomalies Recovered from Geos-3 Altimeter
Data Using Least Squares Collocation"

Abstract

Geos-3 altimeter data provided by the Wallops Flight Center has been processed using techniques developed by Rummel to determine consistent geoid height information without need for a precise Geos-3 orbit. Data was selected from fifty-three processed arcs to determine three 5° equal area anomalies, two $2^\circ \times 2^\circ$ anomalies and two $1^\circ \times 1^\circ$ anomalies. These blocks were located so as to maximize the amount of altimeter data in and surrounding the block whose anomaly was to be estimated. The anomalies were estimated using the technique of least-squares collocation using both an ellipsoidal reference field and a reference field defined by the GEM 7 potential coefficients taken to degree 12. The predicted anomalies showed good agreement with estimates based on terrestrial gravity data. The average predicted standard deviations were ± 3 mgals for the 5° blocks; ± 5 mgals for the $2^\circ \times 2^\circ$ blocks; and ± 10 mgals for the $1^\circ \times 1^\circ$ blocks.

Introduction

For several years discussions have taken place on the recovery of gravity anomalies from satellite altimeter data. These discussions have been limited to simulation studies or studies using data other than altimeter data. In a paper presented last year at the American Geophysical Union Meeting, I described a series of simulation studies that showed the expected accuracy of the recovery of various mean gravity anomalies from altimeter data postulated to be of varying density and of various accuracy. At this time we now have sufficient real altimeter data from Geos-3 to carry out actual anomaly recovery with the results being compared to actual estimates based on terrestrial gravity measurements. This paper reviews the theory, the data used, the results and gives recommendations for future work.

The Theory

The general method that we use is called least-squares collocation (Moritz, 1972). The application of this general method to the use of altimeter data for anomaly recovery work has been discussed by Smith (1974) and Rapp (1974) among others.

We will use the simplified model of collocation where we assume that the observations do not contain systematic components that are described by a term of the form AX where X is a vector of parameters of unknowns. In this case the basic equation of least-squares collocation is:

$$(1) \quad \underline{x} = \underline{s}' + \underline{n}$$

where:

\underline{x} is a measurement (which in our case is a geoid undulation derived from the measured altimeter data);

\underline{s}' is the random signal part of \underline{x} ;

\underline{n} is the random noise of the observation.

One result of least squares collocation can be the estimation of a signal, \underline{s} , of a quantity that we desire to determine, which in our case is gravity anomalies, Δg . The collocation solution of (1) for a predicted signal is:

$$(2) \quad \underline{s} = C_{ss'} (C_{s's'} + C_{nn})^{-1} \underline{x}$$

where:

$C_{ss'}$ is the covariance between the signal to be estimated and the signal component of the measurement;

$C_{s's'}$ is the covariance between the signal components of the measured data;

C_{nn} is the error variance-covariance of the measurements.

The variance-covariance matrix of the predicted signal is:

$$(3) \quad E_{ss} = C_{ss} - C_{ss'}(C_{s's'} + C_{nn})^{-1} C_{s's}$$

where C_{ss} is the covariance matrix for the signals being estimated.

For the prediction of a single gravity anomaly at point P from a set of altimeter data, a , we can write equations (2) and (3) in the following form:

$$(4) \quad \Delta g_p = C_{\epsilon_p a} (C_{aa} + C_{nn})^{-1} \underline{a}$$

$$(5) \quad m_p^2 = C_{\epsilon_p \epsilon_p} - C_{\epsilon_p a} (C_{aa} + C_{nn})^{-1} C_{a \epsilon_p}$$

where m_p is the standard deviation of the prediction and $C_{\epsilon_p \epsilon_p}$ is the expected mean square value of the anomaly block being estimated.

The Covariances

In order to evaluate equations (4) and (5) it is necessary to compute the covariances between the altimeter (undulation) data, as well as the cross-covariance between the anomaly and undulation data. One might consider the use of covariances derived from actual data in the area of computation. The determination of such covariances requires sufficient data which in most cases is lacking. We consequently use the concept of a global covariance function which is isotropic and stationary and which can be computed analytically once certain parameters of an anomaly degree variance model are specified. The details of the formulation of such models and their derivations leading to a computer subroutine that can be used for the generation of the needed covariances are given in Tscherning and Rapp (1974). All covariances used in this paper were computed with subroutine COVA given in that report.

The covariances that are given by COVA are regarded as covariances between point quantities. However, we are estimating mean anomalies, not point anomalies so that special consideration must be given to derive the appropriate covariance value for final use in equations (4) and (5). Thus, for example, with Δg_p considered a mean anomaly, $C_{\epsilon_p a}$ would be the covariance between the mean anomaly block and the altimeter measurement (generally regarded as a point measurement). These mean covariances or variances can be computed by the numerical integration of the point covariance functions over the block in question (Heiskanen and Moritz, 1967, p.277).

The Reference Field

Anomalies or other gravimetric dependent quantities can be given with respect to an ellipsoid reference field or to a higher degree field defined by a set of potential coefficients given to some specified degree, N . It might be expected that more accurate predictions could be carried out by using a high degree reference field but this expectation must be judged against the accuracy of that reference field.

In this paper, for comparison purposes, we will carry out predictions based on the usual ellipsoidal model, and then with a reference model defined by the GEM 7 potential coefficients (Lerch et als., 1975) taken to degree 12. In all cases the final predicted anomaly is to be given with respect to an ellipsoidal reference model.

To carry out this prediction using the higher degree field we first subtract from our altimeter data (actually the undulation implied by the altimeter data), the undulation implied by the potential coefficients. We then predict the mean anomaly with respect to the reference field using the covariances that have had the reference field components removed. To obtain the final predicted anomaly we then add the anomaly in that block being predicted implied by the potential coefficients adopted for the reference field.

In spherical approximation, equations used for computing the reference undulations and anomalies are;

$$(6) \quad \Delta g_N = G \sum_{n=2}^N (n-1) \sum_{m=0}^n (\bar{C}_{nm}^* \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \bar{\varphi})$$

$$(7) \quad N_N = R \sum_{n=2}^N \sum_{m=0}^n (\bar{C}_{nm}^* \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \bar{P}_{nm}(\sin \bar{\varphi}).$$

where:

$\bar{C}_{nn}, \bar{S}_{nn} \dots$ are fully normalized potential coefficients;

$$\bar{C}_{2,0}^* = \bar{C}_{2,0} - \bar{C}_{2,0}(\tau_{\text{eff}});$$

$$\bar{C}_{4,0}^* = \bar{C}_{4,0} - \bar{C}_{4,0}(\tau_{\text{eff}});$$

$\bar{P}_{nn} \dots$ are the fully normalized associated Legendre functions

$\varphi, \lambda \dots$ geocentric latitude and longitude

$R, G \dots$ average values of the earth's radius and gravity.

The Data and Its Accuracy

The Geos-3 data being analyzed in this study represents data received from the Wallops Flight Center for the time period April 21 through May 20, 1975. Of the data analyzed in this paper 48 arcs were received in February 1976, while 5 arcs were received in October 1975. This data consisted both of the long pulse and short pulse mode with the angular spacing between data points being $0^\circ 12'$ for the long pulse mode and $0^\circ 19'$ for the short pulse mode.

The actual altimeter data, converted to raw undulation data, is contaminated by several errors such as orbit error, altimeter bias etc. It is thus necessary to process the data that we have received to remove such errors and to put the data in a consistent system. This processing is described in a paper by Rummel (1976). The input data to the collocation prediction are the unfiltered geoid undulations obtained by Rummel at each of the altimeter data points. These undulations however, may still have errors (other than observation errors) in them which arise from sea state effects and unmodeled errors in the original processing at Wallops or in the error model adopted by Rummel. These errors are expected to be small.

In order to evaluate the C_{nn} matrix in equation (4) and (5) it is necessary to have an accuracy estimate for each measurement. Such an estimate was available from the data supplied by Wallops. Thus estimates varied from approxi-

mately ± 0.5 m to ± 8.5 m with average values being on the order of 1.2 m (Rummel, 1976). Preliminary anomaly estimations were done assuming uniform accuracies of ± 1 m, an error estimate for each track, and the individual error estimates. The predicted anomalies showed only a slight dependence on the accuracy assumed for the data. Consequently, we assumed that the accuracy of the undulations derived from the altimeter data were those given for each altimeter measurement through the processing procedure at the Wallops Flight Center.

Anomaly Predictions

We chose to estimate three 5° equal area anomalies, two $2^\circ \times 2^\circ$ anomalies, and two $1^\circ \times 1^\circ$ anomalies. The location of these anomalies was chosen to maximize the altimeter tracks in and around the blocks. The location of these blocks and an alphabetic block designator are shown in Figure One, with the block coordinates being given in Table One.

Table One. Coordinates of Blocks Used in the Anomaly Prediction.

Letter Designation	Block Size	ϕ	λ
A	5° E.A.	$45^\circ - 40^\circ$	$299^\circ - 306^\circ$
B	5° E.A.	$40^\circ - 35^\circ$	$291^\circ - 297^\circ$
C	5° E.A.	$35^\circ - 30^\circ$	$295^\circ - 301^\circ$
D	$2^\circ \times 2^\circ$	$40^\circ - 38^\circ$	$291^\circ - 293^\circ$
E	$2^\circ \times 2^\circ$	$34^\circ - 32^\circ$	$296^\circ - 298^\circ$
F	$1^\circ \times 1^\circ$	$39^\circ - 38^\circ$	$291^\circ - 292^\circ$
G	$1^\circ \times 1^\circ$	$34^\circ - 33^\circ$	$296^\circ - 297^\circ$

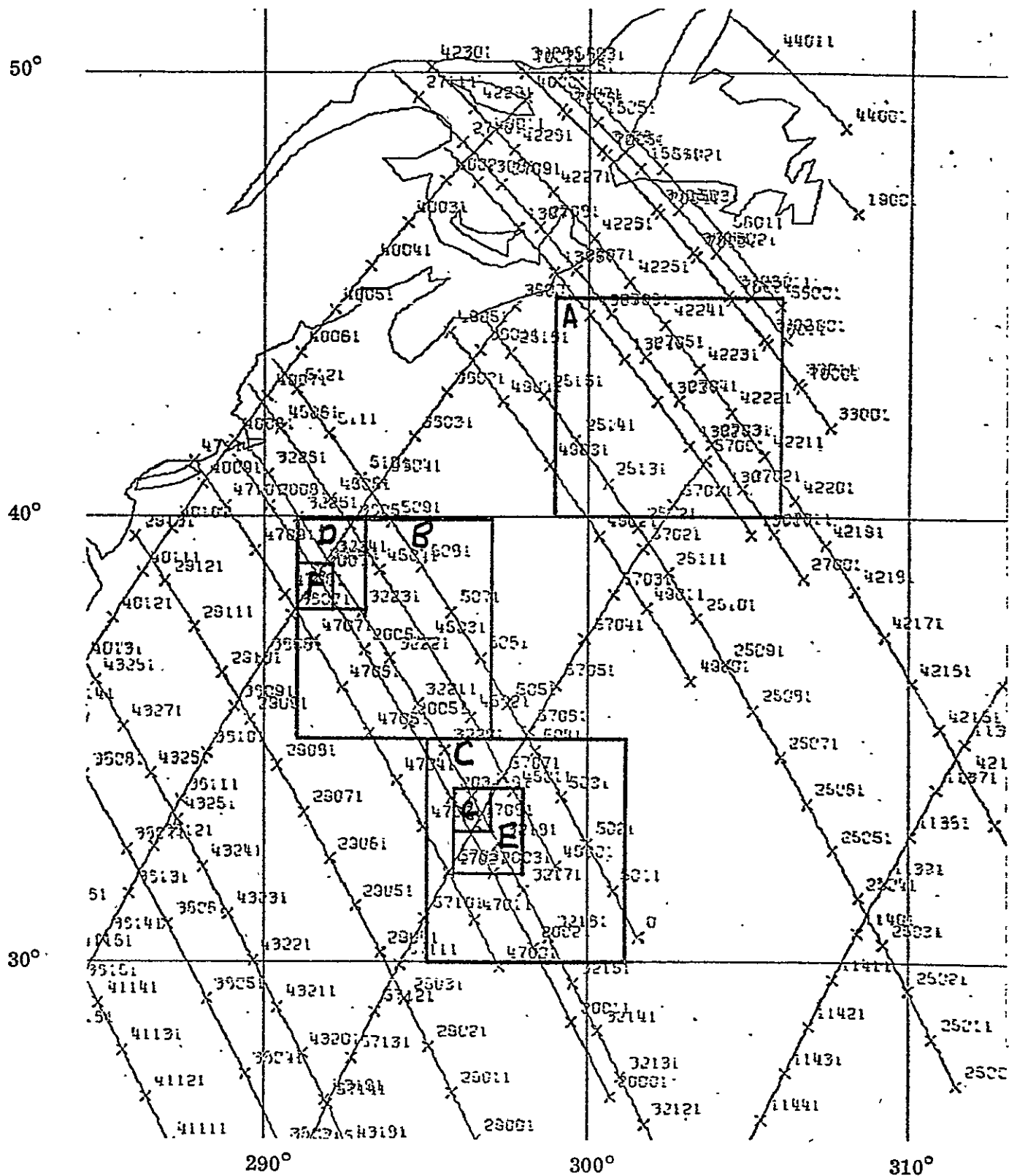


Figure One. Location of Test Blocks and Related Altimeter Tracks

The gravity data used to obtain the check anomalies was an updated version of the DMA AC (1973) $1^\circ \times 1^\circ$ data. This updated version is described in Rapp (1975). The $1^\circ \times 1^\circ$ data is taken directly from the tape. The $2^\circ \times 2^\circ$ and 5° equal area anomalies were formed by a direct mean of the $1^\circ \times 1^\circ$ anomalies in the block. The anomalies and their standard deviations are given in Table Two.

In carrying out the actual predictions several decisions have to be made. Some of these areas of concern are:

1. Reference Model

We have previously discussed the use of an ellipsoidal reference model and a model defined by the GEM 7 coefficients taken to degree 12. Computations have been made with both fields to compare results. Other reference fields and different maximum degrees could have been tested but were not for this paper.

2. Covariances

The covariances used were the global covariances obtained from COVA as previously explained. No optimization of the covariances was done to fit the data in the test area. This may be considered in the future.

3. Data Cap Size

Data within a cap about the center of the block being predicted is to be chosen. The cap is chosen large enough so that data outside this cap does not influence the predicted anomaly or its accuracy within some specific tolerance level. Simulation studies indicated cap sizes on the order of 5° to 7° were reasonable when dealing with an ellipsoidal reference model while a cap size on the order of 3° to 4° was reasonable when a degree 12 field was used. In this paper results for both cap sizes will be shown for the $2^\circ \times 2^\circ$ and $1^\circ \times 1^\circ$ predictions.

4. Data Density

Although the use of all known data points within the chosen cap might be considered for use in the anomaly prediction there are two reasons for

not using all data. First, the size of the matrix to be inverted in (4) and (5) is equal to the number of observations. Thus a large number of observations will lead to large matrices with potentially large inversion times.

Second, the use of many points, close together, may yield some instability in the inversion as the elements of C_{xx} will become similar.

We thus must choose our observations with some care. In computations made for this paper we selected only a certain number of points along a track that fell within the specified cap. We did tests choosing every 2nd point, 3rd point and in some cases 4th point. In some cases the choice of every other point led to some instability relative to the case of using every 3rd point. For the predictions presented in this paper every third point along a track was selected. Results for other point selections are available but they are not presented here.

Results

The anomaly predictions are shown in Table Two. In the case of the use of the reference field to degree 12, the standard deviation of the predicted anomaly has been given assuming the reference field is known perfectly.

Table Two. Anomaly Prediction Results.

Block	Size	Terrestrial Anomaly (mgals)	Predicted E*	Anomaly N=12†	Num. of Points
A	5° E.A.	-17 ± 3	-16 ± 6	-20 ± 3	198
B	5° E.A.	-24 ± 2	-24 ± 5	-25 ± 3	225
C	5° E.A.	- 8 ± 3	- 3 ± 6	-11 ± 3	164
D	2° x 2°	-31 ± 7	-33 ± 7	-32 ± 5	161
E	2° x 2°	- 8 ± 7	- 2 ± 7	- 8 ± 5	142
F	1° x 1°	-25 ± 14	-30 ± 11	-29 ± 9	165
G	1° x 1°	- 8 ± 15	- 3 ± 10	- 9 ± 8	145

* Prediction made with the ellipsoidal reference model.

† Prediction made with GEM 7 to degree 12 as the reference model.

9/1/76

With the exception of one or two cases the results using the ellipsoidal and GEM 7 reference models agree quite well. In addition the predicted anomalies agree well with the terrestrial estimates. For those tests using two different cap sizes the larger cap generally gave the better result.

Conclusions

The results obtained in this paper indicate that terrestrial anomalies can be recovered from the Geos-3 altimeter data using least squares collocation. The predicted values agree with the known values within the range of the predicted standard deviations. These standard deviations for 5° anomalies was about ± 6 mgal which were on the order of ± 3 mgal in ideal circumstances. Nevertheless, a good percentage of existing oceanic 5° equal area anomalies are not known to ± 7 mgal so that the altimeter data can provide a real improvement to their determination.

The predicted accuracy of the $1^\circ \times 1^\circ$ anomalies was on the order of ± 10 mgals (although the actual predictions were better than the standard deviation implied). Denser data should yield accuracies on the order of ± 5 mgals.

Finally we note that these results indicate that the processing of the semi-raw altimeter data yields undulation information of high quality without the need for precise satellite orbits.

Future work on anomaly determination awaits the release of more Geos-3 data.

Acknowledgement

This study was supported, in part, through NASA Contract NAS6-2484, Wallops Flight Center, Wallops Island, Virginia 23337.

Funds for certain computer work was made available through the Instruction and Research Computer Center of The Ohio State University.

References

- DMA AC, 1° x 1° Mean Free-Air Gravity Anomalies, Defense Mapping Agency Aerospace Center, Ref. Publication 73-0002, Dec. 1973.
- Heiskanen, W., and Moritz, H., Physical Geodesy, W. H. Freeman, San Francisco, 1967.
- Lerch, F., Richardson, J., Brown, J., Improvement in the Gravitational Potential Derived from Satellite Data, abstract, EOS, Trans. Amer. Geophys. Union, Vol. 56, No. 6, p. 349, June, 1975.
- Moritz, H., Advanced Least-Squares Methods, Report No. 175, Department of Geodetic Science, The Ohio State University, Columbus, 1972.
- Rapp, R. H., Gravity Anomaly Recovery from Satellite Altimetry Data Using Least Squares Collocation Techniques, Report No. 220, Department of Geodetic Science, The Ohio State University, Columbus, 1974.
- Rapp, R. H., Report of SSG-5-36, Earth Models, paper presented at the International Association of Geodesy Assembly, Grenoble, France, 1975.
- Rapp, R. H., The Gravitational Potential of the Earth to Degree 36 from Terrestrial Gravity Data, paper presented at the International Association of Geodesy Assembly, Grenoble, France, 1975.
- Rummel, R., Geos-3 Altimeter Data Processing for Gravity Anomaly Recovery, abstract, EOS, Trans. Amer. Geophys. Union, Vol. 57, No. 4, p. 234, April, 1976.
- Smith, G., Mean Anomaly Prediction from Terrestrial Gravity Data and Satellite Altimeter Data, Report No. 214, Department of Geodetic Science, The Ohio State University, Columbus, 1974.
- Tscherning, C. C., and Rapp, R. H., Closed Covariance Expressions for Gravity Anomalies, Geoid Undulations, and Deflections of the Vertical Implied by Anomaly Degree Variance Models, Report No. 208, Department of Geodetic Science, The Ohio State University, Columbus, 1974.

1. Report No. NASA CR-141442	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle The Determination of Gravity Anomalies from Geoid Heights Using the Inverse Stokes' Formula		5. Report Date August 1978	
		6. Performing Organization Code	
7. Author(s) Reiner Rummel, Lars Sjoberg, Richard Rapp		8. Performing Organization Report No.	
9. Performing Organization Name and Address Ohio State University Research Foundation 1958 Neil Avenue Columbus, OH 43212		10. Work Unit No.	
		11. Contract or Grant No. NAS6-2484	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Wallops Flight Center Wallops Island, VA 23337		13. Type of Report and Period Covered Contractor Report	
		14. Sponsoring Agency Code	
15. Supplementary Notes			
16. Abstract <p>This report contains three papers related to the determination of mean gravity anomalies given geoid height information as could be derived from the GEOS-3 satellite altimeter. The first paper describes the use of the inverse Stokes' formula derived by Molodenskii. This formula is modified for use with discrete data and applied to a test area where limited altimeter data was available. The main disadvantage of this procedure is the lack of a reliable procedure to determine the accuracy of the estimated anomaly.</p> <p>The second paper looks at the use of Fourier transforms for anomaly prediction. This is done by transforming a least squares collocation solution into the frequency domain. The resultant method was tested on a large sample of real data with very promising results. One advantage of this technique is the reduction of computer time over some other methods.</p> <p>The third paper describes the method of least squares collocation as used for anomaly determination. This method was the method chosen for production anomaly recovery. In this paper, however, tests are made with limited data with promising results.</p>			
17. Key Words (Suggested by Author(s)) GEOS-3 gravity anomalies geoid least squares collocation Stokes' formula		18. Distribution Statement Unclassified - unlimited STAR Category - 42,46,48	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 65	22. Price*